DEPARTMENT OF APPLIED STATISTICS
(COMPUTING SECTION)
UNIVERSITY OF LONDON, UNIVERSITY COLLEGE

TRACTS FOR COMPUTERS

EDITED BY KARL PEARSON, F.R.S.

No. I

Tables of the Digamma and Trigamma Functions

BY ELEANOR PAIRMAN, M.A.

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TRACTS FOR COMPUTERS

No. I

PREFATORY NOTE

During the course of the past five years the Department of Applied Statistics in the University of London (University College) has carried out a great deal of computing work of one kind or another bearing on special war problems of a physical character. Its members have been struck by the absence of any simple text-book for the use of computers and still more by the absence of obviously necessary auxiliary tables. The present series of *Tracts for Computers* will endeavour to fill this gap as far as it lies in our power. It will not concern itself with the higher mathematical theory, but solely with the practical difficulties of the computer, or rather such difficulties as we have met with in our own experience. The first tract will be followed not only by others containing recently computed tables or by the republication of old tables at present very inaccessible, but by tracts dealing with interpolation, quadrature, mechanical integration, calculating machines, tabling machines, and bibliographies of memoirs and of tables having special value to the practical computer. In regard to the present tract, giving the values of the digamma and trigamma functions, we should ourselves have been saved many weeks of work had it been in existence four years ago. Further, we believe it will be of help not only in many physical problems, other than those we have had to deal with ourselves, but to the schoolmaster, who grasps the urgent importance of teaching practical mathematics to the modern schoolboy. The table of logarithms is not the only table that a schoolboy should learn to handle. In most modern computing laboratories a table of logarithms is very rarely used—and when used it is generally one to 10 or 14 figures* where multiplications are necessary which exceed the range of the ordinary multiplying machine. Nowadays the schoolboy ought to be practised in computing, and this practice should run parallel with his algebraic work. He should be exercised in the use of tables which are not becoming obsolete

* And where in seeking the antilog. the schoolboy's knowledge of the process is idle!

like the smaller tables of logarithms. He comes at a fairly early stage to the summation of series and he is liable to regard certain series as unsummable because he has not approached them numerically, just as he unfortunately regards certain integrals as unintegrable, because he is not introduced at a quite early stage to graphical, mechanical and numerical methods of quadrature. The present tract covers a very wide class of numerically summable series, and we can conceive no better practice than the schoolmaster could provide for his pupils by teaching them to sum all such series by tabular aid. If the pupil be asked at the same time to compare the result obtained by summing directly 15 to 20 terms of the series set (using tables of logarithms if he likes!), he will have learnt during the process a good deal of the practical value of logarithms, of tests for convergency, of partial fractions, of interpolation and of the value of tabular aids to the computer. He will further have realised that "proportional parts" are neither the sole nor necessarily adequate method of entering a table;—a belief not indeed infrequently found to dominate the post-graduate as well as the schoolboy mind and probably arising from the same limitation of experience—the very words "mathematical tables" being treated as synonymous with the smaller tables of common and trigonometrical logarithms.

K. P.

August 1919.

TABLES OF THE DIGAMMA AND TRIGAMMA FUNCTIONS

$$\frac{d}{dz}\log\Gamma(1+z) \text{ AND } \frac{d^2}{dz^2}\log\Gamma(1+z),$$

TO FACILITATE THE SUMMATION OF SERIES OF THE FORM

$$S=\sum_{i=1}^{\infty}\frac{a_0+a_1 i+a_2 i^2+\ldots+a_{n-2}i^{n-2}}{(p_1 i+q_1)(p_2 i+q_2)\ldots(p_n i+q_n)}.$$

By Eleanor Pairman, M.A.

In work recently undertaken by the Biometric Laboratory on the torsion-flexure of aeroplane propeller blades it was found necessary to sum a large number of series of the form

$$S=\sum_{i=1}^{\infty}\frac{a_0+a_1 i+a_2 i^2+\ldots+a_{n-2}i^{n-2}}{(p_1 i+q_1)(p_2 i+q_2)\ldots(p_n i+q_n)},$$

where the a's, p's and q's are numerical quantities, and any number of pairs of factors in the denominator may be equal. By multiplying numerator and denominator by a suitable numerical factor this can always be thrown into the form

$$S=\sum_{i=1}^{\infty}\frac{b_0+b_1 i+b_2 i^2+\ldots+b_{m+2n-2}i^{m+2n-2}}{(i+r_1)(i+r_2)\ldots(i+r_m)(i+s_1)^2(i+s_2)^2\ldots(i+s_n)^2}.$$

As it stands this series is only very slowly convergent, as many as from 60 to 80 terms being sometimes required to calculate it to eight figure accuracy.

The expression can always be broken up into partial fractions and we have

$$S=\sum_{i=1}^{\infty}\left[\frac{A_1}{i+r_1}+\frac{A_2}{i+r_2}+\ldots+\frac{A_m}{i+r_m}+\frac{B_1}{i+s_1}+\frac{B_2}{i+s_2}+\ldots+\frac{B_n}{i+s_n}\right.$$
$$\left.+\frac{C_1}{(i+s_1)^2}+\frac{C_2}{(i+s_2)^2}+\ldots+\frac{C_n}{(i+s_n)^2}\right],$$

where the A's, B's and C's are readily determined.

We cannot, however, re-arrange the terms in S and sum each of the series $\Sigma \frac{A_1}{i+r_1}, \ldots, \Sigma \frac{B_1}{i+s_1}, \ldots, \Sigma \frac{C_1}{(i+s_1)^2} \ldots$ separately, since a series of the form $\Sigma \frac{1}{i+\alpha}$ is divergent.

Calling the ith term in the series $v(i)$ we have

$$v(i) = \frac{A_1}{i+r_1} + \frac{A_2}{i+r_2} + \ldots + \frac{A_m}{i+r_m} + \frac{B_1}{i+s_1} + \frac{B_2}{i+s_2} + \ldots + \frac{B_n}{i+s_n}$$
$$+ \frac{C_1}{(i+s_1)^2} + \frac{C_2}{(i+s_2)^2} + \ldots + \frac{C_n}{(i+s_n)^2}$$
$$= A_1\left(\frac{1}{i+r_1} - \frac{1}{i}\right) + A_2\left(\frac{1}{i+r_2} - \frac{1}{i}\right) + \ldots + A_m\left(\frac{1}{i+r_m} - \frac{1}{i}\right)$$
$$+ B_1\left(\frac{1}{i+s_1} - \frac{1}{i}\right) + B_2\left(\frac{1}{i+s_2} - \frac{1}{i}\right) + \ldots + B_n\left(\frac{1}{i+s_n} - \frac{1}{i}\right)$$
$$+ \frac{C_1}{(i+s_1)^2} + \frac{C_2}{(i+s_2)^2} + \ldots + \frac{C_n}{(i+s_n)^2}$$
$$+ \frac{1}{i}(A_1 + A_2 + \ldots + A_m + B_1 + B_2 + \ldots + B_n).$$

Also, from the general theory of Partial Fractions we have*

$$A_1 + A_2 + \ldots + A_m + B_1 + B_2 + \ldots + B_n = 0,$$

which gives
$$v(i) = -\frac{A_1 r_1}{i(i+r_1)} - \frac{A_2 r_2}{i(i+r_2)} - \ldots - \frac{A_m r_m}{i(i+r_m)}$$
$$- \frac{B_1 s_1}{i(i+s_1)} - \frac{B_2 s_2}{i(i+s_2)} - \ldots - \frac{B_n s_n}{i(i+s_n)}$$
$$+ \frac{C_1}{(i+s_1)^2} + \frac{C_2}{(i+s_2)^2} + \ldots + \frac{C_n}{(i+s_n)^2}.$$

* If

$$\frac{b_0 + b_1 i + b_2 i^2 + \ldots + b_{m+2n-2} i^{m+2n-2}}{(i+r_1)(i+r_2)\ldots(i+r_m)(i+s_1)^2(i+s_2)^2\ldots(i+s_n)^2}$$
$$\equiv \frac{A_1}{i+r_1} + \ldots + \frac{A_m}{i+r_m} + \frac{B_1}{i+s_1} + \ldots + \frac{B_n}{i+s_n} + \frac{C_1}{(i+s_1)^2} + \ldots + \frac{C_n}{(i+s_n)^2},$$

then

$$b_0 + b_1 i + b_2 i^2 + \ldots + b_{m+2n-2} i^{m+2n-2}$$
$$\equiv A_1(i+r_1)\ldots(i+r_m)(i+s_1)^2\ldots(i+s_n)^2 + \ldots + A_m(i+r_1)\ldots(i+r_{m-1})(i+s_1)^2\ldots(i+s_n)^2$$
$$+ B_1(i+r_1)\ldots(i+r_m)(i+s_1)(i+s_2)^2\ldots(i+s_n)^2 + \ldots$$
$$+ B_n(i+r_1)\ldots(i+r_m)(i+s_1)^2\ldots(i+s_{n-1})^2(i+s_n)^2$$
$$+ C_1(i+r_1)\ldots(i+r_m)(i+s_2)^2\ldots(i+s_n)^2 + \ldots + C_n(i+r_1)\ldots(i+r_m)(i+s_1)^2\ldots(i+s_{n-1})^2.$$

Equating the coefficients of x^{m+2n-1} on each side of this identity we obtain the result

$$A_1 + A_2 + \ldots + A_m + B_1 + B_2 + \ldots + B_n = 0.$$

But $\sum\limits_{i=1}^{\infty} \frac{1}{i(i+a)}$ and $\sum\limits_{i=1}^{\infty} \frac{1}{(i+b)^2}$ are both absolutely convergent series. Hence we can now re-arrange the terms of S and have

$$
\begin{aligned}
S = -A_1 \sum_{i=1}^{\infty} \frac{r_1}{i(i+r_1)} - A_2 \sum_{i=1}^{\infty} \frac{r_2}{i(i+r_2)} - \dots - A_m \sum_{i=1}^{\infty} \frac{r_m}{i(i+r_m)} \\
- B_1 \sum_{i=1}^{\infty} \frac{s_1}{i(i+s_1)} - B_2 \sum_{i=1}^{\infty} \frac{s_2}{i(i+s_2)} - \dots - B_n \sum_{i=1}^{\infty} \frac{s_n}{i(i+s_n)} \\
- C_1 \sum_{i=1}^{\infty} \frac{1}{(i+s_1)^2} - C_2 \sum_{i=1}^{\infty} \frac{1}{(i+s_2)^2} - \dots - C_n \sum_{i=1}^{\infty} \frac{1}{(i+s_n)^2}.
\end{aligned}
$$

Now it is known* that

$$
\begin{aligned}
\frac{d}{dt} \log \Gamma(1+t) &= -\gamma + \sum_{n=1}^{\infty} \frac{t}{n(n+t)}, \\
\frac{d^2}{dt^2} \log \Gamma(1+t) &= \sum_{n=1}^{\infty} \frac{1}{(n+t)^2},
\end{aligned}
$$

where $\gamma =$ Euler's Constant $= 0{\cdot}5772157\dots$.

Therefore

$$
\begin{aligned}
S = -A_1 \left\{\frac{d}{dr_1} \log \Gamma(1+r_1) + \gamma\right\} - A_2 \left\{\frac{d}{dr_2} \log \Gamma(1+r_2) + \gamma\right\} - \dots \\
- A_m \left\{\frac{d}{dr_m} \log \Gamma(1+r_m) + \gamma\right\} \\
- B_1 \left\{\frac{d}{ds_1} \log \Gamma(1+s_1) + \gamma\right\} - B_2 \left\{\frac{d}{ds_2} \log \Gamma(1+s_2) + \gamma\right\} - \dots \\
- B_n \left\{\frac{d}{ds_n} \log \Gamma(1+s_n) + \gamma\right\} \\
+ C_1 \frac{d^2}{ds_1^{\,2}} \log \Gamma(1+s_1) + C_2 \frac{d^2}{ds_2^{\,2}} \log \Gamma(1+s_2) + \dots + C_n \frac{d^2}{ds_n^{\,2}} \log \Gamma(1+s_n);
\end{aligned}
$$

or putting
$$
\begin{aligned}
\frac{d}{dr_1} \log \Gamma(1+r_1) &= \digamma(r_1), \\
\frac{d^2}{dr_1^{\,2}} \log \Gamma(1+r_1) &= \digamma\!\!\digamma(r_1)\dagger,
\end{aligned}
$$

and remembering that $A_1 + \dots + A_m + B_1 + \dots + B_n = 0$, we have finally

$$
\begin{aligned}
S = -A_1 \digamma(r_1) - A_2 \digamma(r_2) - \dots - A_m \digamma(r_m) \\
- B_1 \digamma(s_1) - B_2 \digamma(s_2) - \dots - B_n \digamma(s_n) \\
+ C_1 \digamma\!\!\digamma(s_1) + C_2 \digamma\!\!\digamma(s_2) + \dots + C_n \digamma\!\!\digamma(s_n),
\end{aligned}
$$

* Cf. Whittaker and Watson, *Modern Analysis* (2nd Edition), § 12·16.

† It is convenient to speak of these functions as the "digamma and trigamma functions."

where $\mathsf{F}(x)$ *and* $\boldsymbol{\mathsf{F}}(x)$ *are the functions here tabled for values of* x *from* $x = 0{\cdot}00$ *to* $x = 20{\cdot}00$, *the increment in* x *being* $\cdot 02$.

The tables were calculated by means of the asymptotic expansions for the logarithmic derivates of the Gamma-function

$$\mathsf{F}(z) = \frac{d}{dz}\log\Gamma(1+z) = \log(1+z) - \frac{1}{2}\frac{1}{z+1} + \sum_{r=1}^{\infty}\frac{(-)^r B_r}{2r\,(1+z)^{2r}},$$

$$\boldsymbol{\mathsf{F}}(z) = \frac{d^2}{dz^2}\log\Gamma(1+z) = \frac{1}{1+z} + \frac{1}{2}\frac{1}{(z+1)^2} + \sum_{r=1}^{\infty}\frac{(-)^{r+1} B_r}{(1+z)^{2r+1}},$$

in conjunction with the formulae

$$\mathsf{F}(z-1) = -\frac{1}{z} + \mathsf{F}(z),$$

$$\boldsymbol{\mathsf{F}}(z-1) = \frac{1}{z^2} + \boldsymbol{\mathsf{F}}(z).$$

If values of the functions outside the range of the tables are desired they may be obtained from the first of these pairs of equations, or, alternatively, from the formulae

$$\mathsf{F}(nz) = \log_e n + \frac{1}{n}\left\{\mathsf{F}\left(z - \frac{n-1}{n}\right) + \mathsf{F}\left(z - \frac{n-2}{n}\right) + \ldots + \mathsf{F}\left(z - \frac{1}{n}\right) + \mathsf{F}(z)\right\}$$

and

$$\boldsymbol{\mathsf{F}}(nz) = \frac{1}{n^2}\left\{\boldsymbol{\mathsf{F}}\left(z - \frac{n-1}{n}\right) + \boldsymbol{\mathsf{F}}\left(z - \frac{n-2}{n}\right) + \ldots + \boldsymbol{\mathsf{F}}\left(z - \frac{1}{n}\right) + \boldsymbol{\mathsf{F}}(z)\right\},$$

which are deduced from Gauss' multiplication-theorem for the Gamma-function.

This latter pair are the more useful for values of x lying between 20 and 40.

E.g.

$$\begin{aligned}
\mathsf{F}(37{\cdot}68) &= \log_e 2 + \tfrac{1}{2}\{\mathsf{F}(18{\cdot}34) + \mathsf{F}(18{\cdot}84)\} \\
&= \quad \cdot 69314718 \\
&\quad + 1{\cdot}46804980 \\
&\quad + 1{\cdot}48114342 \\
&= \quad 3{\cdot}64244040
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\mathsf{F}}(37{\cdot}68) &= \tfrac{1}{4}\{\boldsymbol{\mathsf{F}}(18{\cdot}34) + \boldsymbol{\mathsf{F}}(18{\cdot}84)\} \\
&= \tfrac{1}{4}\left\{\begin{array}{l} \quad \cdot 05306611 \\ + \cdot 05169480 \end{array}\right\} \\
&= \tfrac{1}{4}\{\cdot 10476091\} \\
&= \quad \cdot 02619023.
\end{aligned}$$

Besides the values of the functions there are tabled their second and fourth differences, and the formula of interpolation to be used is Everitt's Central Difference Formula:

$$f(a+\theta\omega)=\theta f_1+\phi f_0-\frac{\theta\phi}{6}\{(\theta+1)\,\delta^2 f_1+(\phi+1)\,\delta^2 f_0\}$$
$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\theta+2)\,\delta^4 f_1+(\phi+2)\,\delta^4 f_0\};$$

where

$$\omega=\text{increment in } x=\cdot 02,$$
$$f_1=f(a+\cdot 02),$$
$$f_0=f(a),$$
$$\theta+\phi=1.$$

E.g. $\digamma(8{\cdot}615)$,

$a=8{\cdot}60.\qquad \theta=\frac{3}{4}.\qquad \phi=\frac{1}{4}.$

$\digamma(8{\cdot}615)$

$$=\ \tfrac{1}{4}(2{\cdot}20877652)+\tfrac{3}{4}(2{\cdot}21096971)+\frac{\frac{3}{4}\cdot\frac{1}{4}}{6}\left\{\begin{array}{l}\tfrac{7}{4}(479)\\+\tfrac{5}{4}(482)\end{array}\right\}$$
$$=\ 2{\cdot}21042186.$$

$\digamma(8{\cdot}615)$

$$=\ \tfrac{1}{4}(\cdot 10977999)+\tfrac{3}{4}(\cdot 10953972)-\frac{\frac{3}{4}\cdot\frac{1}{4}}{6}\left\{\begin{array}{l}\tfrac{5}{4}(106)\\+\tfrac{7}{4}(105)\end{array}\right\}$$
$$=\ \cdot 10959969.$$

It was not discovered until the present tables were almost completed that a certain portion of the work had been already performed. Gauss (*Werke*, Bd III, pp. 161, 162, Göttingen, 1863 etc.) gives tables of $\frac{d}{dz}\log\Gamma(1+z)$ correct to 18 decimal places for values of z between 0 and 1, the increment in z being ·01. Prof. G. N. Watson (*Report of the British Association*, 1916, pp. 125, 126) gives tables of the same function correct to 13 decimal places for all integers and halves of odd integers from 0 to 100. These two valuable tables have been made use of as a check on the present 8-figure tables.

So far as has been discovered there are no other tables of $\frac{d^2}{dz^2}\log\Gamma(1+z)$.

The following illustrations are some which have recently occurred in practice. They were previously calculated by the longer method, and the results obtained from the two methods agree exactly.

(i)
$$S=\sum_{i=1}^{\infty}\frac{1}{(4i+3)(4i+2)\,4i\,(4i+4)}$$
$$=\sum_{i=1}^{\infty}\left[\frac{1}{3}\frac{1}{4i+3}-\frac{1}{4}\frac{1}{4i+2}+\frac{1}{24}\frac{1}{4i}-\frac{1}{8}\frac{1}{4i+4}\right]$$
$$=\sum_{i=1}^{\infty}\left[\frac{1}{12}\frac{1}{i+\cdot 75}-\frac{1}{16}\frac{1}{i+\cdot 50}+\frac{1}{96}\frac{1}{i}-\frac{1}{32}\frac{1}{i+1{\cdot}00}\right]$$
$$=-\tfrac{1}{12}\digamma(\cdot 75)+\tfrac{1}{16}\digamma(\cdot 50)-\tfrac{1}{96}\digamma(\cdot 00)+\tfrac{1}{32}\digamma(1{\cdot}00).$$

$$\mathrm{F}(\cdot 75) = \tfrac{1}{2}\left\{\begin{matrix} \cdot 23980321 \\ + \cdot 25508551 \end{matrix}\right\} + \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}}{6}\left\{\begin{matrix} 22802 \\ + 22161 \end{matrix}\right\} - \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{120}\left\{\begin{matrix} 27 \\ + 26 \end{matrix}\right\}$$

$$= \cdot 24744436 + 2810 - 1 = \cdot 24747245.$$

Hence $S = \cdot 00228062 - \cdot 02062270$

$$\begin{array}{l} 601266 \\ 1321201 \\ \hline \end{array}$$

$$= \cdot 02150529 - \cdot 02062270 = \cdot 00088259.$$

(ii) $$S = \sum_{i=1}^{\infty} \frac{1}{(4i+2)(4i+1)(4i+3)^2}$$

$$= \sum_{i=1}^{\infty}\left[-\frac{1}{4i+2} + \frac{1}{4}\frac{1}{4i+1} + \frac{3}{4}\frac{1}{4i+3} + \frac{1}{2}\frac{1}{(4i+3)^2}\right]$$

$$= \tfrac{1}{4}\,\mathrm{F}(\cdot 50) - \tfrac{1}{16}\,\mathrm{F}(\cdot 25) - \tfrac{3}{16}\,\mathrm{F}(\cdot 75) + \tfrac{1}{32}\,\mathrm{F}(\cdot 75).$$

$$\mathrm{F}(\cdot 25) = -\tfrac{1}{2}\left\{\begin{matrix} \cdot 23949368 \\ + \cdot 21554617 \end{matrix}\right\} + \tfrac{1}{16}\left\{\begin{matrix} 54251 \\ + 52024 \end{matrix}\right\} - \tfrac{3}{256}\left\{\begin{matrix} 140 \\ + 129 \end{matrix}\right\}$$

$$= -\cdot 22751993 + 6642 - 3 = -\cdot 22745354.$$

$$\mathrm{F}(\cdot 75) = \tfrac{1}{2}\left\{\begin{matrix} \cdot 23980321 \\ + \cdot 25508551 \end{matrix}\right\} + \tfrac{1}{16}\left\{\begin{matrix} 22802 \\ + 22161 \end{matrix}\right\} - \tfrac{3}{256}\left\{\begin{matrix} 27 \\ + 26 \end{matrix}\right\}$$

$$= \cdot 24744436 + 2810 - 1 = \cdot 24747245.$$

$$\mathrm{F}(\cdot 75) = \tfrac{1}{2}\left\{\begin{matrix} \cdot 76976109 \\ + \cdot 75852269 \end{matrix}\right\} - \tfrac{1}{16}\left\{\begin{matrix} 32695 \\ 31363 \end{matrix}\right\} + \tfrac{3}{256}\left\{\begin{matrix} 75 \\ + 70 \end{matrix}\right\}$$

$$= \cdot 76414189 - 4004 + 2 = \cdot 76410187.$$

Hence $S = \cdot 00912249 - \cdot 04640108$

$$\begin{array}{l} \cdot 01421585 \\ \cdot 02387818 \\ \hline \end{array}$$

$$= \cdot 04721652 - \cdot 04640108 = \cdot 00081544.$$

(iii) $$S = \sum_{i=1}^{\infty} \frac{30}{(60i+29)(60i+30)(60i+31)^2}$$

$$= \sum_{i=1}^{\infty}\left[\frac{30}{4}\frac{1}{60i+29} - \frac{30}{60i+30} + \frac{90}{4}\frac{1}{60i+31} + \frac{30}{2}\frac{1}{(60i+31)^2}\right]$$

$$= -\tfrac{1}{8}\,\mathrm{F}\left(\tfrac{29}{60}\right) + \tfrac{1}{2}\,\mathrm{F}\left(\tfrac{30}{60}\right) - \tfrac{3}{8}\,\mathrm{F}\left(\tfrac{31}{60}\right) + \tfrac{1}{240}\,\mathrm{F}\left(\tfrac{31}{60}\right).$$

$$\mathrm{F}\left(\tfrac{29}{60}\right) = \mathrm{F}\left(\cdot 48 + \tfrac{1}{6}(\cdot 02)\right)$$

$$= \begin{matrix} \tfrac{5}{6}(\cdot 01762627) \\ + \tfrac{1}{6}(\cdot 03648997) \end{matrix} + \frac{\frac{1}{6}\cdot\frac{5}{6}}{6}\left\{\begin{matrix} \tfrac{11}{6}(34312) \\ + \tfrac{7}{6}(33157) \end{matrix}\right\} - \frac{\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{11}{6}}{120}\left\{\begin{matrix} \tfrac{17}{6}(60) \\ + \tfrac{13}{6}(55) \end{matrix}\right\}$$

$$= \cdot 02077022 + 2352 - 1 = \cdot 02079373.$$

$$\mathsf{F}\left(\tfrac{31}{60}\right)=\mathsf{F}\left(\cdot 50+\tfrac{5}{6}(\cdot 02)\right)$$

$$=\begin{matrix}\tfrac{1}{6}(\cdot 03648997)+\\ +\tfrac{5}{6}(\cdot 05502211)\end{matrix}\;\frac{\tfrac{1}{6}\cdot\tfrac{5}{6}}{6}\left\{\begin{matrix}\tfrac{7}{6}(33157)\\ +\tfrac{11}{6}(32056)\end{matrix}\right\}-\frac{\tfrac{1}{6}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{6}}{120}\left\{\begin{matrix}\tfrac{13}{6}(55)\\ +\tfrac{17}{6}(52)\end{matrix}\right\}$$

$$=\cdot 05193342+2256-1=\cdot 05195597.$$

$$\mathbf{F}\left(\tfrac{31}{60}\right)=\mathbf{F}\left(\cdot 50+\tfrac{5}{6}(\cdot 02)\right)$$

$$=\begin{matrix}\tfrac{1}{6}(\cdot 93480220)-\\ +\tfrac{5}{6}(\cdot 91850353)\end{matrix}\;\frac{\tfrac{1}{6}\cdot\tfrac{5}{6}}{6}\left\{\begin{matrix}\tfrac{7}{6}(56379)\\ +\tfrac{11}{6}(53685)\end{matrix}\right\}-\frac{\tfrac{1}{6}\cdot\tfrac{5}{6}\cdot\tfrac{7}{6}\cdot\tfrac{11}{6}}{120}\left\{\begin{matrix}\tfrac{13}{6}(178)\\ +\tfrac{17}{6}(165)\end{matrix}\right\}$$

$$=\cdot 92121998-3801+2=\cdot 92118199.$$

Hence

$$\begin{array}{rl} S= & \cdot 01824499-\cdot 00259922\\ & \;\;\;\;383826\;\;\;\;\;\;\;\;\;1948349\\ = & \cdot 02208325-\cdot 02208271=\cdot 00000054.\end{array}$$

(iv) $$S=\sum_{i=1}^{\infty}\frac{1}{(2i+1)(6i-7)(6i+13)^2}$$

$$=\sum_{i=1}^{\infty}\left[-\frac{1}{10^3}\frac{1}{2i+1}+\frac{3}{4}\frac{1}{10^3}\frac{1}{6i-7}+\frac{9}{4}\frac{1}{10^3}\frac{1}{6i+13}+\frac{3}{2}\frac{1}{10^2}\frac{1}{(6i+13)^2}\right]$$

$$=\frac{1}{10^3}\left[\frac{1}{2}\mathsf{F}(\cdot 50)-\frac{1}{8}\mathsf{F}\left(-\frac{7}{6}\right)-\frac{3}{8}\mathsf{F}\left(\frac{13}{6}\right)+\frac{5}{12}\mathbf{F}\left(\frac{13}{6}\right)\right].$$

$$\mathsf{F}\left(-\tfrac{7}{6}\right)=\tfrac{6}{7}+\mathsf{F}\left(-\tfrac{1}{6}\right)=\tfrac{6}{7}+\tfrac{6}{1}+\mathsf{F}\left(\tfrac{5}{6}\right)$$

$$=6\tfrac{6}{7}+\mathsf{F}\left(\cdot 82+\tfrac{2}{3}(\cdot 02)\right)$$

$$=\begin{matrix}6\cdot 85714286+\tfrac{1}{3}(\cdot 29962710)+\\ +\tfrac{2}{3}(\cdot 31405886)\end{matrix}\;\frac{\tfrac{1}{3}\cdot\tfrac{2}{3}}{6}\left\{\begin{matrix}\tfrac{4}{3}(20390)\\ +\tfrac{5}{3}(19845)\end{matrix}\right\}-\frac{\tfrac{1}{3}\cdot\tfrac{2}{3}\cdot\tfrac{4}{3}\cdot\tfrac{5}{3}}{120}\left\{\begin{matrix}\tfrac{7}{3}(21)\\ +\tfrac{8}{3}(20)\end{matrix}\right\}$$

$$=6\cdot 85714286+\cdot 30924827+2232=7\cdot 16641345.$$

$$\mathsf{F}\left(\tfrac{13}{6}\right)=\mathsf{F}\left(2\cdot 16+\tfrac{1}{3}(\cdot 02)\right)$$

$$=\begin{matrix}\tfrac{2}{3}(\cdot 98407878)+\\ +\tfrac{1}{3}(\cdot 99148578)\end{matrix}\;\frac{\tfrac{1}{3}\cdot\tfrac{2}{3}}{6}\left\{\begin{matrix}\tfrac{5}{3}(5468)\\ +\tfrac{4}{3}(5389)\end{matrix}\right\}$$

$$=\cdot 98654778+604=\cdot 98655382.$$

$$\mathbf{F}\left(\tfrac{13}{6}\right)=\mathbf{F}\left(2\cdot 16+\tfrac{1}{3}(\cdot 02)\right)$$

$$=\begin{matrix}\tfrac{2}{3}(\cdot 37171057)-\\ +\tfrac{1}{3}(\cdot 36899640)\end{matrix}\;\frac{\tfrac{1}{3}\cdot\tfrac{2}{3}}{6}\left\{\begin{matrix}\tfrac{5}{3}(3982)\\ +\tfrac{4}{3}(3897)\end{matrix}\right\}$$

$$=\cdot 37080585-438=\cdot 37080147.$$

Hence $$S=\frac{1}{10^3}\begin{bmatrix}\cdot 01824499-\cdot 89580168\\ \cdot 15450061\;\;\;\;\;\cdot 36995768\end{bmatrix}$$

$$=\frac{1}{10^3}[\cdot 17274560-1\cdot 26575936]=-\cdot 00109301.$$

1—5

x	F(x)	δ^2 −	δ^4 −	Ƒ(x)	δ^2 +	δ^4 +	x	F(x)	δ^2 −	Ƒ(x)	δ^2 +	δ^4 +
·00	−·57721566	96198	399	1·64493407	259920	1959	1·00	·42278434	16166	·64493407	19760	34
·02	−·54478931	91192	362	1·59811819	240933	1741	1·02	·43560285	15777	·63694941	19068	31
·04	−·51327488	86549	329	1·55371164	223686	1551	1·04	·44826358	15403	·62915543	18406	30
·06	−·48262594	82234	300	1·51154195	207990	1385	1·06	·46077029	15041	·62154551	17775	28
·08	−·45279934	78220	273	1·47145216	193679	1240	1·08	·47312659	14692	·61411334	17171	27
·10	−·42375494	74479	251	1·43329915	180608	1111	1·10	·48533597	14354	·60685287	16594	25
·12	−·39545533	70988	229	1·39695222	168648	999	1·12	·49740181	14028	·59975834	16042	24
·14	−·36786561	67727	210	1·36229177	157686	899	1·14	·50932737	13712	·59282424	15514	23
·16	−·34095315	64675	193	1·32920817	147624	811	1·16	·52111581	13407	·58604527	15009	21
·18	−·31468744	61816	178	1·29760082	138373	733	1·18	·53277018	13112	·57941640	14525	20
·20	−·28903990	59135	164	1·26737721	129855	663	1·20	·54429344	12826	·57293276	14061	19
·22	−·26398370	56618	151	1·23845214	122000	602	1·22	·55568843	12549	·56658973	13616	18
·24	−·23949368	54251	140	1·21074707	114747	546	1·24	·56695793	12281	·56038287	13190	17
·26	−·21554617	52024	129	1·18418948	108041	497	1·26	·57810463	12021	·55430790	12781	17
·28	−·19211890	49926	120	1·15871230	101832	453	1·28	·58913110	11770	·54836074	12388	16
·30	−·16919089	47948	111	1·13425343	96075	413	1·30	·60003988	11526	·54253746	12012	15
·32	−·14674236	46080	103	1·11075532	90732	378	1·32	·61083340	11289	·53683430	11650	14
·34	−·12475463	44316	95	1·08816454	85767	346	1·34	·62151403	11060	·53124763	11302	14
·36	−·10321006	42647	89	1·06643142	81148	317	1·36	·63208406	10837	·52577398	10968	13
·38	−·08209196	41068	83	1·04550978	76845	291	1·38	·64254572	10621	·52041001	10646	12
·40	−·06138454	39572	78	1·02535659	72833	267	1·40	·65290117	10411	·51515251	10337	12
·42	−·04107284	38153	71	1·00593172	69088	246	1·42	·66315251	10207	·50999838	10040	11
·44	−·02114267	36806	68	·98719773	65588	226	1·44	·67330177	10009	·50494465	9754	11
·46	−·00158056	35528	63	·96911962	62315	209	1·46	·68335094	9817	·49998845	9478	10
·48	+·01762627	34312	60	·95166466	59250	193	1·48	·69330194	9630	·49512704	9213	10
·50	·03648997	33157	55	·93480220	56379	178	1·50	·70315664	9449	·49035776	8957	9
·52	·05502211	32056	52	·91850353	53685	165	1·52	·71291685	9272	·48567804	8711	9
·54	·07323369	31008	49	·90274170	51156	152	1·54	·72258434	9100	·48108543	8473	9
·56	·09113519	30009	46	·88749142	48779	142	1·56	·73216083	8933	·47657755	8244	8
·58	·10873660	29056	43	·87272894	46544	131	1·58	·74164799	8770	·47215211	8023	8
·60	·12604745	28146	41	·85843189	44440	122	1·60	·75104745	8612	·46780689	7809	7
·62	·14307684	27277	39	·84457925	42458	113	1·62	·76036079	8458	·46353977	7603	7
·64	·15983345	26447	36	·83115118	40590	106	1·64	·76958955	8308	·45934868	7405	7
·66	·17632559	25653	34	·81812901	38827	98	1·66	·77873523	8162	·45523164	7213	7
·68	·19256120	24893	33	·80549511	37162	92	1·68	·78779930	8019	·45118672	7027	6
·70	·20854787	24166	30	·79323283	35590	85	1·70	·79678317	7881	·44721207	6848	6
·72	·22429289	23469	29	·78132645	34102	80	1·72	·80568824	7745	·44330590	6674	6
·74	·23980321	22802	27	·76976109	32695	75	1·74	·81451585	7614	·43946647	6507	5
·76	·25508551	22161	26	·75852269	31363	70	1·76	·82326733	7485	·43569211	6345	5
·78	·27014620	21546	25	·74759791	30100	65	1·78	·83194396	7360	·43198120	6188	5
·80	·28499143	20957	23	·73697414	28903	61	1·80	·84054699	7238	·42833216	6036	5
·82	·29962710	20390	22	·72663940	27768	57	1·82	·84907765	7118	·42474349	5889	5
·84	·31405886	19845	21	·71658233	26690	54	1·84	·85753712	7002	·42121371	5747	5
·86	·32829217	19322	20	·70679216	25665	51	1·86	·86592658	6888	·41774141	5610	4
·88	·34233226	18819	19	·69725865	24692	48	1·88	·87424715	6777	·41432520	5476	4
·90	·35618416	18334	18	·68797206	23766	45	1·90	·88249995	6669	·41096375	5347	4
·92	·36985272	17868	17	·67892313	22886	42	1·92	·89068606	6564	·40765577	5222	4
·94	·38334261	17418	16	·67010306	22047	40	1·94	·89880653	6460	·40440001	5100	4
·96	·39665832	16985	16	·66150345	21248	37	1·96	·90686240	6359	·40119525	4983	4
·98	·40980417	16568	15	·65311633	20487	35	1·98	·91485467	6261	·39804031	4869	3
1·00	·42278434	16166		·64493407	19760	34	2·00	·92278434	6165	·39493407	4758	

Interpolation Formula:

$$f(a+\cdot 02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\} \text{ where } \theta+\phi=1.$$

x	F (x)	δ^2 −	Ƒ (x)	δ^2 +	x	F (x)	δ^2 −	Ƒ (x)	δ^2 +
2·00	·92278434	6165	·39493407	4758	3·00	1·25611767	3201	·28382296	1795
2·02	·93065235	6071	·39187540	4650	3·02	1·26177818	3166	·28223108	1765
2·04	·93845966	5979	·38886323	4546	3·04	1·26740703	3131	·28065686	1736
2·06	·94620718	5889	·38589653	4445	3·06	1·27300457	3097	·27910000	1708
2·08	·95389582	5801	·38297428	4347	3·08	1·27857114	3063	·27756022	1680
2·10	·96152644	5715	·38009550	4252	3·10	1·28410709	3029	·27603723	1653
2·12	·96909992	5631	·37725923	4159	3·12	1·28961274	2997	·27453077	1626
2·14	·97661709	5549	·37446456	4069	3·14	1·29508843	2964	·27304056	1600
2·16	·98407878	5468	·37171057	3982	3·16	1·30053447	2933	·27156636	1574
2·18	·99148578	5389	·36899640	3897	3·18	1·30595119	2901	·27010790	1550
2·20	·99883889	5312	·36632119	3814	3·20	1·31133889	2871	·26866494	1525
2·22	1·00613888	5237	·36368413	3734	3·22	1·31669789	2840	·26723723	1501
2·24	1·01338651	5163	·36108440	3656	3·24	1·32202848	2811	·26582453	1478
2·26	1·02058250	5090	·35852123	3580	3·26	1·32733097	2781	·26442661	1455
2·28	1·02772759	5020	·35599385	3506	3·28	1·33260564	2752	·26304323	1432
2·30	1·03482249	4950	·35350154	3434	3·30	1·33785279	2724	·26167418	1410
2·32	1·04186788	4882	·35104357	3364	3·32	1·34307270	2696	·26031923	1389
2·34	1·04886446	4816	·34861924	3296	3·34	1·34826565	2668	·25897817	1368
2·36	1·05581287	4750	·34622788	3230	3·36	1·35343192	2641	·25765078	1347
2·38	1·06271379	4686	·34386881	3166	3·38	1·35857177	2615	·25633686	1327
2·40	1·06956784	4624	·34154140	3103	3·40	1·36368548	2588	·25503621	1307
2·42	1·07637565	4562	·33924501	3042	3·42	1·36877331	2562	·25374862	1287
2·44	1·08313784	4502	·33697905	2982	3·44	1·37383551	2537	·25247391	1268
2·46	1·08985501	4443	·33474290	2924	3·46	1·37887235	2512	·25121187	1249
2·48	1·09652775	4385	·33253599	2867	3·48	1·38388407	2487	·24996233	1231
2·50	1·10315664	4328	·33035776	2812	3·50	1·38887093	2462	·24872510	1213
2·52	1·10974225	4273	·32820765	2759	3·52	1·39383316	2438	·24750000	1195
2·54	1·11628513	4218	·32608512	2706	3·54	1·39877101	2414	·24628685	1178
2·56	1·12278583	4164	·32398966	2655	3·56	1·40368471	2391	·24508548	1161
2·58	1·12924489	4112	·32192075	2605	3·58	1·40857450	2368	·24389572	1144
2·60	1·13566284	4060	·31987790	2557	3·60	1·41344062	2345	·24271741	1128
2·62	1·14204018	4009	·31786061	2510	3·62	1·41828327	2323	·24155037	1112
2·64	1·14837743	3960	·31586842	2463	3·64	1·42310270	2301	·24039445	1096
2·66	1·15467508	3911	·31390087	2418	3·66	1·42789913	2279	·23924949	1081
2·68	1·16093363	3863	·31195749	2374	3·68	1·43267276	2258	·23811534	1065
2·70	1·16715354	3816	·31003786	2331	3·70	1·43742381	2236	·23699184	1051
2·72	1·17333529	3770	·30814154	2289	3·72	1·44215250	2216	·23587885	1036
2·74	1·17947935	3724	·30626811	2248	3·74	1·44685903	2195	·23477622	1022
2·76	1·18558617	3680	·30441717	2208	3·76	1·45154362	2175	·23368380	1008
2·78	1·19165619	3636	·30258831	2169	3·78	1·45620645	2155	·23260146	994
2·80	1·19768985	3593	·30078114	2131	3·80	1·46084774	2135	·23152906	980
2·82	1·20368758	3551	·29899529	2094	3·82	1·46546768	2115	·23046647	967
2·84	1·20964980	3509	·29723038	2058	3·84	1·47006646	2096	·22941354	954
2·86	1·21557693	3468	·29548604	2022	3·86	1·47464428	2077	·22837014	941
2·88	1·22146937	3428	·29376193	1987	3·88	1·47920133	2059	·22733616	928
2·90	1·22732754	3389	·29205768	1953	3·90	1·48373779	2040	·22631146	916
2·92	1·23315181	3350	·29037298	1920	3·92	1·48825385	2022	·22529592	904
2·94	1·23894258	3312	·28870747	1888	3·94	1·49274969	2004	·22428942	892
2·96	1·24470024	3275	·28706084	1856	3·96	1·49722549	1986	·22329184	880
2·98	1·25042514	3238	·28543277	1825	3·98	1·50168142	1969	·22230305	869
3·00	1·25611767	3201	·28382296	1795	4·00	1·50611767	1952	·22132296	857

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi\,(\theta+1)\,(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\ \text{ where }\ \theta+\phi=1.$$

x	F (x)	δ^2 −	Ƒ (x)	δ^2 +	x	F (x)	δ^2 −	Ƒ (x)	δ^2 +
4·00	1·50611767	1952	·22132296	857	5·00	1·70611767	1312	·18132296	473
4·02	1·51053440	1935	·22035143	846	5·02	1·70973759	1302	·18066952	468
4·04	1·51493178	1918	·21938836	835	5·04	1·71334448	1293	·18002076	463
4·06	1·51930999	1901	·21843364	824	5·06	1·71693845	1284	·17937663	458
4·08	1·52366918	1885	·21748717	814	5·08	1·72051958	1275	·17873709	453
4·10	1·52800953	1869	·21654883	803	5·10	1·72408796	1266	·17810208	448
4·12	1·53233119	1853	·21561852	793	5·12	1·72764369	1257	·17747155	444
4·14	1·53663432	1837	·21469614	783	5·14	1·73118685	1248	·17684546	439
4·16	1·54091909	1821	·21378160	773	5·16	1·73471754	1239	·17622376	435
4·18	1·54518564	1806	·21287478	763	5·18	1·73823583	1230	·17560640	430
4·20	1·54943413	1791	·21197560	754	5·20	1·74174182	1222	·17499335	425
4·22	1·55366471	1776	·21108395	744	5·22	1·74523559	1213	·17438455	421
4·24	1·55787754	1761	·21019975	735	5·24	1·74871723	1205	·17377996	417
4·26	1·56207275	1746	·20932290	726	5·26	1·75218682	1197	·17317954	412
4·28	1·56625050	1732	·20845331	717	5·28	1·75564444	1188	·17258324	408
4·30	1·57041093	1718	·20759089	708	5·30	1·75909018	1180	·17199103	404
4·32	1·57455418	1704	·20673555	700	5·32	1·76252411	1172	·17140286	400
4·34	1·57868040	1690	·20588721	691	5·34	1·76594632	1164	·17081869	396
4·36	1·58278972	1676	·20504578	683	5·36	1·76935688	1156	·17023847	392
4·38	1·58688228	1662	·20421117	674	5·38	1·77275588	1149	·16966218	388
4·40	1·59095821	1649	·20338331	666	5·40	1·77614340	1141	·16908976	384
4·42	1·59501765	1636	·20256212	658	5·42	1·77951950	1133	·16852119	380
4·44	1·59906074	1623	·20174751	650	5·44	1·78288427	1126	·16795642	376
4·46	1·60308760	1610	·20093940	643	5·46	1·78623778	1118	·16739541	373
4·48	1·60709836	1597	·20013772	635	5·48	1·78958011	1111	·16683813	369
4·50	1·61109315	1584	·19934239	628	5·50	1·79291133	1104	·16628454	365
4·52	1·61507210	1572	·19855333	620	5·52	1·79623152	1096	·16573460	362
4·54	1·61903532	1560	·19777048	613	5·54	1·79954074	1089	·16518828	358
4·56	1·62298296	1547	·19699376	606	5·56	1·80283907	1082	·16464554	355
4·58	1·62691511	1535	·19622310	599	5·58	1·80612658	1075	·16410635	351
4·60	1·63083192	1523	·19545843	592	5·60	1·80940335	1068	·16357067	348
4·62	1·63473349	1512	·19469967	585	5·62	1·81266943	1061	·16303847	344
4·64	1·63861995	1500	·19394677	578	5·64	1·81592491	1054	·16250971	341
4·66	1·64249140	1488	·19319964	572	5·66	1·81916985	1047	·16198437	338
4·68	1·64634797	1477	·19245824	565	5·68	1·82240431	1041	·16146240	335
4·70	1·65018977	1466	·19172249	559	5·70	1·82562836	1034	·16094378	331
4·72	1·65401691	1455	·19099232	552	5·72	1·82884208	1027	·16042848	328
4·74	1·65782950	1444	·19026768	546	5·74	1·83204552	1021	·15991646	325
4·76	1·66162765	1433	·18954850	540	5·76	1·83523876	1014	·15940768	322
4·78	1·66541147	1422	·18883472	534	5·78	1·83842185	1008	·15890213	319
4·80	1·66918107	1412	·18812629	528	5·80	1·84159487	1002	·15839977	316
4·82	1·67293656	1401	·18742313	522	5·82	1·84475787	995	·15790057	313
4·84	1·67667803	1391	·18672520	516	5·84	1·84791091	989	·15740450	310
4·86	1·68040560	1380	·18603243	511	5·86	1·85105407	983	·15691153	307
4·88	1·68411937	1370	·18534476	505	5·88	1·85418739	977	·15642163	304
4·90	1·68781943	1360	·18466215	500	5·90	1·85731095	971	·15593477	302
4·92	1·69150588	1350	·18398454	494	5·92	1·86042480	965	·15545093	299
4·94	1·69517884	1340	·18331186	489	5·94	1·86352901	959	·15497008	296
4·96	1·69883839	1331	·18264408	483	5·96	1·86662363	953	·15449219	293
4·98	1·70248464	1321	·18198112	478	5·98	1·86970872	947	·15401723	291
5·00	1·70611767	1312	·18132296	473	6·00	1·87278434	941	·15354518	288

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi\,(\theta+1)\,(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\quad\text{where }\theta+\phi=1.$$

x	F (x)	δ^2 −	Ꞙ (x)	δ^2 +	x	F (x)	δ^2 −	Ꞙ (x)	δ^2 +
6·00	1·87278434	941	·15354518	288	7·00	2·01564148	708	·13313701	188
6·02	1·87585054	935	·15307600	285	7·02	2·01830068	704	·13278396	186
6·04	1·87890739	930	·15260968	283	7·04	2·02095285	700	·13243277	185
6·06	1·88195495	924	·15214619	280	7·06	2·02359801	697	·13208343	184
6·08	1·88499326	919	·15168550	278	7·08	2·02623620	693	·13173593	182
6·10	1·88802239	913	·15122758	275	7·10	2·02886746	690	·13139025	181
6·12	1·89104238	908	·15077242	273	7·12	2·03149182	686	·13104637	179
6·14	1·89405330	902	·15031998	270	7·14	2·03410932	682	·13070429	178
6·16	1·89705520	897	·14987024	268	7·16	2·03672000	679	·13036398	177
6·18	1·90004813	891	·14942318	265	7·18	2·03932390	675	·13002545	175
6·20	1·90303214	886	·14897878	263	7·20	2·04192103	672	·12968866	174
6·22	1·90600730	881	·14853701	261	7·22	2·04451145	668	·12935361	172
6·24	1·90897364	876	·14809784	258	7·24	2·04709519	665	·12902028	171
6·26	1·91193123	871	·14766126	256	7·26	2·04967228	662	·12868867	170
6·28	1·91488011	865	·14722725	254	7·28	2·05224275	658	·12835875	168
6·30	1·91782034	860	·14679577	252	7·30	2·05480664	655	·12803052	167
6·32	1·92075196	855	·14636681	250	7·32	2·05736398	651	·12770396	166
6·34	1·92367502	851	·14594034	247	7·34	2·05991481	648	·12737906	165
6·36	1·92658959	846	·14551635	245	7·36	2·06245915	645	·12705581	163
6·38	1·92949569	841	·14509481	243	7·38	2·06499705	642	·12673419	162
6·40	1·93239340	836	·14467570	241	7·40	2·06752853	638	·12641419	161
6·42	1·93528274	831	·14425900	239	7·42	2·07005363	635	·12609581	160
6·44	1·93816377	826	·14384469	237	7·44	2·07257237	632	·12577902	159
6·46	1·94103654	822	·14343275	235	7·46	2·07508480	629	·12546382	157
6·48	1·94390110	817	·14302316	233	7·48	2·07759094	626	·12515019	156
6·50	1·94675748	812	·14261590	231	7·50	2·08009082	623	·12483812	155
6·52	1·94960575	808	·14221094	229	7·52	2·08258447	619	·12452760	154
6·54	1·95244594	803	·14180828	227	7·54	2·08507193	616	·12421862	153
6·56	1·95527810	799	·14140788	225	7·56	2·08755323	613	·12391117	152
6·58	1·95810227	794	·14100974	223	7·58	2·09002839	610	·12360524	151
6·60	1·96091850	790	·14061383	221	7·60	2·09249745	607	·12330081	149
6·62	1·96372684	785	·14022013	219	7·62	2·09496043	604	·12299788	148
6·64	1·96652732	781	·13982863	218	7·64	2·09741737	601	·12269642	147
6·66	1·96932000	776	·13943930	216	7·66	2·09986830	598	·12239644	146
6·68	1·97210491	772	·13905214	214	7·68	2·10231324	596	·12209793	145
6·70	1·97488209	768	·13866711	212	7·70	2·10475222	593	·12180086	144
6·72	1·97765160	764	·13828420	211	7·72	2·10718528	590	·12150523	143
6·74	1·98041348	760	·13790341	209	7·74	2·10961244	587	·12121103	142
6·76	1·98316775	755	·13752470	207	7·76	2·11203373	584	·12091826	141
6·78	1·98591448	751	·13714806	205	7·78	2·11444918	581	·12062689	140
6·80	1·98865369	747	·13677347	204	7·80	2·11685882	579	·12033692	139
6·82	1·99138543	743	·13640093	202	7·82	2·11926267	576	·12004834	138
6·84	1·99410974	739	·13603040	200	7·84	2·12166076	573	·11976114	137
6·86	1·99682666	735	·13566188	199	7·86	2·12405312	570	·11947530	136
6·88	1·99953623	731	·13529534	197	7·88	2·12643978	568	·11919083	135
6·90	2·00223849	727	·13493078	196	7·90	2·12882077	565	·11890771	134
6·92	2·00493347	723	·13456818	194	7·92	2·13119610	562	·11862593	133
6·94	2·00762123	719	·13420752	193	7·94	2·13356581	560	·11834548	132
6·96	2·01030179	716	·13384878	191	7·96	2·13592993	557	·11806635	131
6·98	2·01297519	712	·13349195	190	7·98	2·13828847	554	·11778853	130
7·00	2·01564148	708	·13313701	188	8·00	2·14064148	552	·11751201	129

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\quad\text{where }\theta+\phi=1.$$

x	F (x)	δ^2 −	F (x)	δ^2 +	x	F (x)	δ^2 −	F (x)	δ^2 +
8·00	2·14064148	552	·11751201	129	9·00	2·25175259	442	·10516634	93
8·02	2·14298896	549	·11723679	128	9·02	2·25385371	440	·10494580	92
8·04	2·14533096	547	·11696286	128	9·04	2·25595043	438	·10472619	92
8·06	2·14766749	544	·11669020	127	9·06	2·25804276	436	·10450750	91
8·08	2·14999857	542	·11641880	126	9·08	2·26013073	435	·10428971	90
8·10	2·15232425	539	·11614867	125	9·10	2·26221436	433	·10407283	90
8·12	2·15464453	537	·11587978	124	9·12	2·26429365	431	·10385685	89
8·14	2·15695945	534	·11561214	123	9·14	2·26636864	429	·10364176	89
8·16	2·15926902	532	·11534572	122	9·16	2·26843933	427	·10342757	88
8·18	2·16157328	529	·11508053	121	9·18	2·27050575	426	·10321425	88
8·20	2·16387225	527	·11481656	121	9·20	2·27256790	424	·10300181	87
8·22	2·16616595	524	·11455379	120	9·22	2·27462582	422	·10279025	87
8·24	2·16845441	522	·11429222	119	9·24	2·27667952	421	·10257955	86
8·26	2·17073765	519	·11403184	118	9·26	2·27872901	419	·10236971	86
8·28	2·17301569	517	·11377265	117	9·28	2·28077431	417	·10216073	85
8·30	2·17528856	515	·11351463	117	9·30	2·28281545	415	·10195260	85
8·32	2·17755629	512	·11325777	116	9·32	2·28485242	414	·10174531	84
8·34	2·17981888	510	·11300208	115	9·34	2·28688526	412	·10153887	84
8·36	2·18207638	508	·11274753	114	9·36	2·28891398	410	·10133326	83
8·38	2·18432879	506	·11249413	114	9·38	2·29093860	409	·10112848	82
8·40	2·18657615	503	·11224186	113	9·40	2·29295913	407	·10092452	82
8·42	2·18881847	501	·11199072	112	9·42	2·29497559	405	·10072139	81
8·44	2·19105579	499	·11174070	111	9·44	2·29698799	404	·10051907	81
8·46	2·19328811	497	·11149179	111	9·46	2·29899635	402	·10031756	81
8·48	2·19551547	494	·11124399	110	9·48	2·30100070	401	·10011686	80
8·50	2·19773788	492	·11099729	109	9·50	2·30300103	399	·09991696	80
8·52	2·19995536	490	·11075168	108	9·52	2·30499738	397	·09971785	80
8·54	2·20216795	488	·11050715	108	9·54	2·30698975	396	·09951954	79
8·56	2·20437566	486	·11026369	107	9·56	2·30897817	394	·09932201	78
8·58	2·20657851	484	·11002131	106	9·58	2·31096264	393	·09912526	78
8·60	2·20877652	482	·10977999	106	9·60	2·31294318	391	·09892929	77
8·62	2·21096971	479	·10953972	105	9·62	2·31491982	390	·09873410	77
8·64	2·21315811	477	·10930050	104	9·64	2·31689255	388	·09853967	76
8·66	2·21534174	475	·10906233	103	9·66	2·31886141	387	·09834601	76
8·68	2·21752061	473	·10882519	103	9·68	2·32082640	385	·09815310	75
8·70	2·21969475	471	·10858907	102	9·70	2·32278754	384	·09796095	75
8·72	2·22186418	469	·10835398	101	9·72	2·32474484	382	·09776955	75
8·74	2·22402892	467	·10811990	101	9·74	2·32669832	381	·09757890	74
8·76	2·22618899	465	·10788683	100	9·76	2·32864800	379	·09738898	74
8·78	2·22834440	463	·10765477	100	9·78	2·33059389	378	·09719981	73
8·80	2·23049518	461	·10742369	99	9·80	2·33253600	376	·09701137	73
8·82	2·23264135	459	·10719361	98	9·82	2·33447435	375	·09682365	72
8·84	2·23478293	457	·10696451	98	9·84	2·33640895	373	·09663666	72
8·86	2·23691994	455	·10673639	97	9·86	2·33833982	372	·09645039	72
8·88	2·23905240	453	·10650923	96	9·88	2·34026697	370	·09626484	71
8·90	2·24118032	451	·10628304	96	9·90	2·34219042	369	·09608000	71
8·92	2·24330372	450	·10605781	95	9·92	2·34411018	368	·09589587	70
8·94	2·24542264	448	·10583353	95	9·94	2·34602626	366	·09571244	70
8·96	2·24753707	446	·10561019	94	9·96	2·34793868	365	·09552971	70
8·98	2·24964705	444	·10538780	93	9·98	2·34984745	363	·09534768	69
9·00	2·25175259	442	·10516634	93	10·00	2·35175259	362	·09516634	69

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi\,(\theta+1)\,(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\ \text{where } \theta+\phi=1.$$

x	F (x)	δ^2 −	Ғ (x)	δ^2 +	x	F (x)	δ^2 −	Ғ (x)	δ^2 +
10·00	2·35175259	362	·09516634	69	11·00	2·44266168	302	·08690187	52
10·02	2·35365411	361	·09498568	68	11·02	2·44439821	301	·08675119	52
10·04	2·35555202	359	·09480571	68	11·04	2·44613173	300	·08660103	52
10·06	2·35744634	358	·09462642	68	11·06	2·44786225	299	·08645139	52
10·08	2·35933708	357	·09444781	67	11·08	2·44958979	298	·08630226	51
10·10	2·36122426	355	·09426987	67	11·10	2·45131435	297	·08615365	51
10·12	2·36310788	354	·09409260	67	11·12	2·45303594	296	·08600554	51
10·14	2·36498797	353	·09391599	66	11·14	2·45475457	295	·08585795	51
10·16	2·36686453	351	·09374005	66	11·16	2·45647026	294	·08571086	50
10·18	2·36873757	350	·09356476	65	11·18	2·45818301	293	·08556427	50
10·20	2·37060712	349	·09339013	65	11·20	2·45989283	292	·08541819	50
10·22	2·37247318	347	·09321614	65	11·22	2·46159974	291	·08527260	50
10·24	2·37433577	346	·09304280	64	11·24	2·46330374	290	·08512750	49
10·26	2·37619490	345	·09287011	64	11·26	2·46500485	289	·08498290	49
10·28	2·37805058	343	·09269806	64	11·28	2·46670306	288	·08483879	49
10·30	2·37990283	342	·09252664	63	11·30	2·46839840	287	·08469517	49
10·32	2·38175165	341	·09235585	63	11·32	2·47009087	286	·08455203	48
10·34	2·38359706	340	·09218569	63	11·34	2·47178048	285	·08440938	48
10·36	2·38543908	338	·09201616	62	11·36	2·47346725	284	·08426720	48
10·38	2·38727771	337	·09184725	62	11·38	2·47515118	283	·08412551	48
10·40	2·38911297	336	·09167896	61	11·40	2·47683227	282	·08398428	47
10·42	2·39094488	335	·09151128	61	11·42	2·47851055	281	·08384354	47
10·44	2·39277343	334	·09134422	61	11·44	2·48018602	280	·08370326	47
10·46	2·39459865	332	·09117776	60	11·46	2·48185868	279	·08356345	47
10·48	2·39642054	331	·09101191	60	11·48	2·48352856	278	·08342410	46
10·50	2·39823913	330	·09084666	60	11·50	2·48519565	277	·08328522	46
10·52	2·40005442	329	·09068201	60	11·52	2·48685997	276	·08314681	46
10·54	2·40186641	328	·09051796	59	11·54	2·48852153	275	·08300885	46
10·56	2·40367514	326	·09035449	59	11·56	2·49018033	275	·08287134	45
10·58	2·40548060	325	·09019162	59	11·58	2·49183638	274	·08273430	45
10·60	2·40728281	324	·09002933	58	11·60	2·49348970	273	·08259770	45
10·62	2·40908177	323	·08986762	58	11·62	2·49514029	272	·08246155	45
10·64	2·41087751	322	·08970650	58	11·64	2·49678817	271	·08232586	45
10·66	2·41267004	321	·08954595	57	11·66	2·49843333	270	·08219060	44
10·68	2·41445936	319	·08938597	57	11·68	2·50007579	269	·08205580	44
10·70	2·41624548	318	·08922656	57	11·70	2·50171557	268	·08192143	44
10·72	2·41802842	317	·08906772	56	11·72	2·50335265	267	·08178750	44
10·74	2·41980819	316	·08890945	56	11·74	2·50498707	267	·08165401	43
10·76	2·42158480	315	·08875173	56	11·76	2·50661882	266	·08152095	43
10·78	2·42335827	314	·08859458	56	11·78	2·50824791	265	·08138833	43
10·80	2·42512859	313	·08843798	55	11·80	2·50987435	264	·08125613	43
10·82	2·42689579	312	·08828193	55	11·82	2·51149816	263	·08112437	43
10·84	2·42865987	310	·08812643	55	11·84	2·51311933	262	·08099303	42
10·86	2·43042085	309	·08797148	54	11·86	2·51473788	261	·08086212	42
10·88	2·43217874	308	·08781707	54	11·88	2·51635382	261	·08073162	42
10·90	2·43393354	307	·08766320	54	11·90	2·51796715	260	·08060155	42
10·92	2·43568527	306	·08750987	54	11·92	2·51957788	259	·08047190	42
10·94	2·43743394	305	·08735708	53	11·94	2·52118603	258	·08034266	41
10·96	2·43917955	304	·08720481	53	11·96	2·52279159	257	·08021384	41
10·98	2·44092213	303	·08705308	53	11·98	2·52439459	256	·08008543	41
11·00	2·44266168	302	·08690187	52	12·00	2·52599501	256	·07995743	41

Interpolation Formula:

$$f(a+·02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\text{ where }\theta+\phi=1.$$

x	F (x)	δ^2 −	F (x)	δ^2 +	x	F (x)	δ^2 −	F (x)	δ^2 +
12·00	2·52599501	256	·07995743	41	13·00	2·60291809	219	·07404027	32
12·02	2·52759289	255	·07982984	41	13·02	2·60439780	219	·07393084	32
12·04	2·52918821	254	·07970265	40	13·04	2·60587533	218	·07382174	32
12·06	2·53078099	253	·07957587	40	13·06	2·60735067	217	·07371295	32
12·08	2·53237125	252	·07944949	40	13·08	2·60882385	217	·07360449	32
12·10	2·53395898	252	·07932351	40	13·10	2·61029485	216	·07349635	32
12·12	2·53554419	251	·07919793	40	13·12	2·61176370	215	·07338852	32
12·14	2·53712690	250	·07907275	39	13·14	2·61323040	215	·07328101	31
12·16	2·53870710	249	·07894796	39	13·16	2·61469494	214	·07317381	31
12·18	2·54028482	248	·07882357	39	13·18	2·61615735	214	·07306693	31
12·20	2·54186005	248	·07869956	39	13·20	2·61761762	213	·07296035	31
12·22	2·54343280	247	·07857595	39	13·22	2·61907577	212	·07285409	31
12·24	2·54500309	246	·07845272	39	13·24	2·62053179	212	·07274814	31
12·26	2·54657091	245	·07832988	38	13·26	2·62198570	211	·07264249	31
12·28	2·54813629	245	·07820742	38	13·28	2·62343749	210	·07253715	30
12·30	2·54969921	244	·07808535	38	13·30	2·62488718	210	·07243212	30
12·32	2·55125970	243	·07796365	38	13·32	2·62633478	209	·07232739	30
12·34	2·55281776	242	·07784234	38	13·34	2·62778028	209	·07222296	30
12·36	2·55437340	242	·07772140	38	13·36	2·62922370	208	·07211883	30
12·38	2·55592662	241	·07760083	37	13·38	2·63066504	207	·07201500	30
12·40	2·55747743	240	·07748064	37	13·40	2·63210430	207	·07191147	30
12·42	2·55902585	239	·07736082	37	13·42	2·63354150	206	·07180824	30
12·44	2·56057187	238	·07724137	37	13·44	2·63497663	206	·07170530	29
12·46	2·56211551	238	·07712229	37	13·46	2·63640971	205	·07160266	29
12·48	2·56365676	237	·07700358	37	13·48	2·63784074	204	·07150031	29
12·50	2·56519565	236	·07688522	36	13·50	2·63926973	204	·07139826	29
12·52	2·56673218	236	·07676724	36	13·52	2·64069667	203	·07129649	29
12·54	2·56826634	235	·07664961	36	13·54	2·64212159	203	·07119501	29
12·56	2·56979816	234	·07653234	36	13·56	2·64354447	202	·07109383	29
12·58	2·57132764	233	·07641544	36	13·58	2·64496534	202	·07099293	29
12·60	2·57285478	233	·07629888	36	13·60	2·64638419	201	·07089231	28
12·62	2·57437960	232	·07618269	35	13·62	2·64780104	200	·07079198	28
12·64	2·57590209	231	·07606684	35	13·64	2·64921587	200	·07069193	28
12·66	2·57742227	231	·07595135	35	13·66	2·65062872	199	·07059217	28
12·68	2·57894015	230	·07583621	35	13·68	2·65203956	199	·07049269	28
12·70	2·58045572	229	·07572142	35	13·70	2·65344842	198	·07039348	28
12·72	2·58196901	229	·07560697	35	13·72	2·65485530	198	·07029456	28
12·74	2·58348000	228	·07549287	34	13·74	2·65626021	197	·07019591	28
12·76	2·58498872	227	·07537911	34	13·76	2·65766314	196	·07009754	28
12·78	2·58649517	226	·07526569	34	13·78	2·65906411	196	·06999944	27
12·80	2·58799935	226	·07515262	34	13·80	2·66046312	195	·06990162	27
12·82	2·58950128	225	·07503988	34	13·82	2·66186018	195	·06980407	27
12·84	2·59100095	224	·07492748	34	13·84	2·66325529	194	·06970679	27
12·86	2·59249838	224	·07481542	33	13·86	2·66464845	194	·06960979	27
12·88	2·59399357	223	·07470369	33	13·88	2·66603968	193	·06951305	27
12·90	2·59548653	222	·07459230	33	13·90	2·66742898	193	·06941658	27
12·92	2·59697726	222	·07448124	33	13·92	2·66881635	192	·06932038	27
12·94	2·59846578	221	·07437050	33	13·94	2·67020179	192	·06922445	27
12·96	2·59995209	220	·07426010	33	13·96	2·67158533	191	·06912878	26
12·98	2·60143619	220	·07415002	33	13·98	2·67296695	191	·06903337	26
13·00	2·60291809	219	·07404027	32	14·00	2·67434666	190	·06893823	26

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\ \text{ where } \theta+\phi=1.$$

x	F (x)	δ^2 −	Ғ (x)	δ^2 +	x	F (x)	δ^2 −	Ғ (x)	δ^2 +
14·00	2·67434666	190	·06893823	26	15·00	2·74101333	166	·06449378	21
14·02	2·67572448	190	·06884335	26	15·02	2·74230237	166	·06441073	21
14·04	2·67710040	189	·06874873	26	15·04	2·74358976	165	·06432789	21
14·06	2·67847443	188	·06865436	26	15·06	2·74487549	165	·06424526	21
14·08	2·67984657	188	·06856026	26	15·08	2·74615957	165	·06416285	21
14·10	2·68121684	187	·06846642	26	15·10	2·74744201	164	·06408065	21
14·12	2·68258523	187	·06837283	26	15·12	2·74872280	164	·06399865	21
14·14	2·68395175	187	·06827950	25	15·14	2·75000195	163	·06391687	21
14·16	2·68531641	186	·06818642	25	15·16	2·75127947	163	·06383529	21
14·18	2·68667921	185	·06809359	25	15·18	2·75255537	163	·06375392	21
14·20	2·68804016	185	·06800102	25	15·20	2·75382963	162	·06367276	21
14·22	2·68939926	184	·06790870	25	15·22	2·75510228	162	·06359181	21
14·24	2·69075651	184	·06781663	25	15·24	2·75637331	161	·06351106	21
14·26	2·69211192	183	·06772480	25	15·26	2·75764272	161	·06343052	20
14·28	2·69346550	183	·06763323	25	15·28	2·75891053	160	·06335018	20
14·30	2·69481725	182	·06754190	25	15·30	2·76017673	160	·06327004	20
14·32	2·69616718	182	·06745082	25	15·32	2·76144133	160	·06319011	20
14·34	2·69751529	181	·06735999	24	15·34	2·76270434	159	·06311037	20
14·36	2·69886158	181	·06726940	24	15·36	2·76396575	159	·06303084	20
14·38	2·70020607	180	·06717905	24	15·38	2·76522557	158	·06295151	20
14·40	2·70154874	180	·06708894	24	15·40	2·76648381	158	·06287238	20
14·42	2·70288962	180	·06699908	24	15·42	2·76774047	158	·06279345	20
14·44	2·70422871	179	·06690945	24	15·44	2·76899555	157	·06271471	20
14·46	2·70556600	179	·06682007	24	15·46	2·77024906	157	·06263617	20
14·48	2·70690151	178	·06673092	24	15·48	2·77150100	156	·06255783	20
14·50	2·70823524	178	·06664201	24	15·50	2·77275137	156	·06247968	20
14·52	2·70956720	177	·06655334	24	15·52	2·77400019	156	·06240173	19
14·54	2·71089738	177	·06646490	23	15·54	2·77524744	155	·06232397	19
14·56	2·71222579	176	·06637670	23	15·56	2·77649315	155	·06224641	19
14·58	2·71355245	176	·06628873	23	15·58	2·77773730	155	·06216904	19
14·60	2·71487734	175	·06620100	23	15·60	2·77897991	154	·06209186	19
14·62	2·71620049	175	·06611350	23	15·62	2·78022098	154	·06201487	19
14·64	2·71752189	174	·06602622	23	15·64	2·78146050	153	·06193807	19
14·66	2·71884154	174	·06593918	23	15·66	2·78269850	153	·06186147	19
14·68	2·72015945	173	·06585237	23	15·68	2·78393496	153	·06178505	19
14·70	2·72147564	173	·06576578	23	15·70	2·78516990	152	·06170882	19
14·72	2·72279009	173	·06567942	23	15·72	2·78640332	152	·06163278	19
14·74	2·72410281	172	·06559329	23	15·74	2·78763522	151	·06155692	19
14·76	2·72541382	172	·06550738	22	15·76	2·78886560	151	·06148126	19
14·78	2·72672311	171	·06542170	22	15·78	2·79009447	151	·06140577	19
14·80	2·72803069	171	·06533624	22	15·80	2·79132183	150	·06133048	18
14·82	2·72933656	170	·06525101	22	15·82	2·79254769	150	·06125536	18
14·84	2·73064073	170	·06516600	22	15·84	2·79377204	150	·06118043	18
14·86	2·73194320	169	·06508121	22	15·86	2·79499491	149	·06110569	18
14·88	2·73324398	169	·06499663	22	15·88	2·79621627	149	·06103112	18
14·90	2·73454307	168	·06491228	22	15·90	2·79743615	149	·06095674	18
14·92	2·73584047	168	·06482815	22	15·92	2·79865454	148	·06088254	18
14·94	2·73713620	168	·06474423	22	15·94	2·79987145	148	·06080852	18
14·96	2·73843024	167	·06466053	22	15·96	2·80108689	147	·06073468	18
14·98	2·73972262	167	·06457705	22	15·98	2·80230084	147	·06066102	18
15·00	2·74101333	166	·06449378	21	16·00	2·80351333	147	·06058753	18

Interpolation Formula:

$$f(a+\cdot 02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\text{ where }\theta+\phi=1.$$

x	F (x)	δ^2 −	Ƒ (x)	δ^2 +	x	F (x)	δ^2 −	Ƒ (x)	δ^2 +
16·00	2·80351333	147	·06058753	18	17·00	2·86233686	131	·05712733	15
16·02	2·80472435	146	·06051423	18	17·02	2·86347875	130	·05706215	15
16·04	2·80593390	146	·06044110	18	17·04	2·86461934	130	·05699712	15
16·06	2·80714199	146	·06036815	18	17·06	2·86575864	130	·05693224	15
16·08	2·80834863	146	·06029537	17	17·08	2·86689664	130	·05686750	15
16·10	2·80955381	145	·06022277	17	17·10	2·86803334	129	·05680291	15
16·12	2·81075754	145	·06015034	17	17·12	2·86916875	129	·05673847	15
16·14	2·81195982	144	·06007809	17	17·14	2·87030288	129	·05667418	15
16·16	2·81316066	144	·06000601	17	17·16	2·87143572	128	·05661003	15
16·18	2·81436006	144	·05993410	17	17·18	2·87256728	128	·05654602	14
16·20	2·81555803	143	·05986237	17	17·20	2·87369756	128	·05648216	14
16·22	2·81675456	143	·05979081	17	17·22	2·87482657	127	·05641845	14
16·24	2·81794966	143	·05971941	17	17·24	2·87595430	127	·05635488	14
16·26	2·81914334	142	·05964819	17	17·26	2·87708076	127	·05629145	14
16·28	2·82033559	142	·05957714	17	17·28	2·87820596	126	·05622816	14
16·30	2·82152642	142	·05950626	17	17·30	2·87932989	126	·05616502	14
16·32	2·82271584	141	·05943554	17	17·32	2·88045256	126	·05610201	14
16·34	2·82390385	141	·05936499	17	17·34	2·88157397	126	·05603915	14
16·36	2·82509044	141	·05929461	17	17·36	2·88269413	125	·05597643	14
16·38	2·82627563	140	·05922440	17	17·38	2·88381303	125	·05591385	14
16·40	2·82745942	140	·05915435	17	17·40	2·88493068	125	·05585141	14
16·42	2·82864181	140	·05908447	17	17·42	2·88604709	124	·05578911	14
16·44	2·82982280	139	·05901476	16	17·44	2·88716225	124	·05572694	14
16·46	2·83100240	139	·05894520	16	17·46	2·88827617	124	·05566492	14
16·48	2·83218061	139	·05887581	16	17·48	2·88938885	124	·05560303	14
16·50	2·83335743	138	·05880659	16	17·50	2·89050029	123	·05554128	14
16·52	2·83453287	138	·05873752	16	17·52	2·89161050	123	·05547967	14
16·54	2·83570693	138	·05866862	16	17·54	2·89271948	123	·05541819	14
16·56	2·83687962	137	·05859988	16	17·56	2·89382723	123	·05535685	14
16·58	2·83805093	137	·05853130	16	17·58	2·89493375	122	·05529565	14
16·60	2·83922087	137	·05846289	16	17·60	2·89603905	122	·05523458	13
16·62	2·84038945	136	·05839463	16	17·62	2·89714314	122	·05517365	13
16·64	2·84155666	136	·05832653	16	17·64	2·89824600	121	·05511285	13
16·66	2·84272251	136	·05825859	16	17·66	2·89934765	121	·05505218	13
16·68	2·84388700	135	·05819080	16	17·68	2·90044809	121	·05499165	13
16·70	2·84505014	135	·05812318	16	17·70	2·90154732	121	·05493125	13
16·72	2·84621193	135	·05805571	16	17·72	2·90264534	120	·05487098	13
16·74	2·84737237	134	·05798840	16	17·74	2·90374216	120	·05481084	13
16·76	2·84853147	134	·05792124	16	17·76	2·90483777	120	·05475084	13
16·78	2·84968922	134	·05785424	15	17·78	2·90593219	120	·05469097	13
16·80	2·85084564	134	·05778739	15	17·80	2·90702541	119	·05463123	13
16·82	2·85200072	133	·05772070	15	17·82	2·90811744	119	·05457161	13
16·84	2·85315447	133	·05765416	15	17·84	2·90920828	119	·05451213	13
16·86	2·85430689	133	·05758778	15	17·86	2·91029793	119	·05445278	13
16·88	2·85545798	132	·05752154	15	17·88	2·91138639	118	·05439356	13
16·90	2·85660775	132	·05745546	15	17·90	2·91247367	118	·05433446	13
16·92	2·85775620	132	·05738954	15	17·92	2·91355977	118	·05427550	13
16·94	2·85890333	131	·05732376	15	17·94	2·91464469	117	·05421666	13
16·96	2·86004915	131	·05725813	15	17·96	2·91572844	117	·05415795	13
16·98	2·86119366	131	·05719265	15	17·98	2·91681101	117	·05409936	13
17·00	2·86233686	131	·05712733	15	18·00	2·91789241	117	·05404091	13

Interpolation Formula:

$$f(a+{\cdot}02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi\,(\theta+1)\,(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\ \text{where}\ \theta+\phi=1.$$

x	F (x)	δ^2 −	₣ (x)	δ^2 +	x	F (x)	δ^2 −	₣ (x)	δ^2 +
18·00	2·91789241	117	·05404091	13	19·00	2·97052399	105	·05127082	11
18·02	2·91897265	117	·05398257	13	19·02	2·97154888	105	·05121831	11
18·04	2·92005172	116	·05392437	13	19·04	2·97257273	105	·05116591	11
18·06	2·92112962	116	·05386629	13	19·06	2·97359552	104	·05111362	11
18·08	2·92220637	116	·05380833	12	19·08	2·97461727	104	·05106143	11
18·10	2·92328196	115	·05375050	12	19·10	2·97563798	104	·05100935	11
18·12	2·92435639	115	·05369280	12	19·12	2·97665765	104	·05095738	11
18·14	2·92542967	115	·05363521	12	19·14	2·97767627	104	·05090551	11
18·16	2·92650180	115	·05357776	12	19·16	2·97869387	103	·05085374	11
18·18	2·92757278	115	·05352042	12	19·18	2·97971042	103	·05080208	10
18·20	2·92864262	114	·05346320	12	19·20	2·98072595	103	·05075053	10
18·22	2·92971131	114	·05340611	12	19·22	2·98174045	103	·05069908	10
18·24	2·93077886	114	·05334914	12	19·24	2·98275391	103	·05064774	10
18·26	2·93184528	114	·05329229	12	19·26	2·98376636	102	·05059650	10
18·28	2·93291056	113	·05323557	12	19·28	2·98477778	102	·05054536	10
18·30	2·93397470	113	·05317896	12	19·30	2·98578817	102	·05049432	10
18·32	2·93503771	113	·05312247	12	19·32	2·98679755	102	·05044339	10
18·34	2·93609960	113	·05306611	12	19·34	2·98780591	102	·05039256	10
18·36	2·93716036	112	·05300986	12	19·36	2·98881325	101	·05034184	10
18·38	2·93822000	112	·05295373	12	19·38	2·98981958	101	·05029121	10
18·40	2·93927851	112	·05289772	12	19·40	2·99082490	101	·05024069	10
18·42	2·94033591	112	·05284183	12	19·42	2·99182921	101	·05019027	10
18·44	2·94139218	111	·05278606	12	19·44	2·99283251	101	·05013995	10
18·46	2·94244735	111	·05273040	12	19·46	2·99383481	100	·05008973	10
18·48	2·94350140	111	·05267486	12	19·48	2·99483610	100	·05003961	10
18·50	2·94455434	111	·05261944	12	19·50	2·99583639	100	·04998959	10
18·52	2·94560618	111	·05256414	12	19·52	2·99683569	100	·04993967	10
18·54	2·94665691	110	·05250895	12	19·54	2·99783398	100	·04988985	10
18·56	2·94770654	110	·05245387	12	19·56	2·99883128	99	·04984013	10
18·58	2·94875507	110	·05239892	12	19·58	2·99982759	99	·04979051	10
18·60	2·94980250	110	·05234407	11	19·60	3·00082290	99	·04974099	10
18·62	2·95084883	109	·05228934	11	19·62	3·00181723	99	·04969157	10
18·64	2·95189407	109	·05223473	11	19·64	3·00281057	98	·04964224	10
18·66	2·95293822	109	·05218023	11	19·66	3·00380292	98	·04959301	10
18·68	2·95398128	109	·05212584	11	19·68	3·00479429	98	·04954388	10
18·70	2·95502325	108	·05207157	11	19·70	3·00578467	98	·04949485	10
18·72	2·95606414	108	·05201741	11	19·72	3·00677408	98	·04944591	10
18·74	2·95710395	108	·05196336	11	19·74	3·00776251	98	·04939707	10
18·76	2·95814268	108	·05190943	11	19·76	3·00874997	97	·04934833	10
18·78	2·95918033	108	·05185560	11	19·78	3·00973645	97	·04929968	10
18·80	2·96021690	107	·05180189	11	19·80	3·01072195	97	·04925113	10
18·82	2·96125240	107	·05174829	11	19·82	3·01170649	97	·04920267	10
18·84	2·96228684	107	·05169480	11	19·84	3·01269006	97	·04915431	9
18·86	2·96332020	107	·05164142	11	19·86	3·01367266	96	·04910605	9
18·88	2·96435249	106	·05158815	11	19·88	3·01465430	96	·04905788	9
18·90	2·96538372	106	·05153499	11	19·90	3·01563498	96	·04900980	9
18·92	2·96641389	106	·05148194	11	19·92	3·01661470	96	·04896182	9
18·94	2·96744300	106	·05142900	11	19·94	3·01759345	96	·04891393	9
18·96	2·96847105	106	·05137616	11	19·96	3·01857125	95	·04886613	9
18·98	2·96949805	105	·05132344	11	19·98	3·01954810	95	·04881843	9
19·00	2·97052399	105	·05127082	11	20·00	3·02052399	95	·04877082	9

Interpolation Formula:

$$f(a+\cdot 02\theta)=\phi f_0+\theta f_1-\frac{\theta\phi}{6}\{(\phi+1)\,\delta^2 f_0+(\theta+1)\,\delta^2 f_1\}$$

$$+\frac{\theta\phi(\theta+1)(\phi+1)}{120}\{(\phi+2)\,\delta^4 f_0+(\theta+2)\,\delta^4 f_1\}\ \text{ where }\ \theta+\phi=1.$$

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TRACTS FOR COMPUTERS

EDITED BY KARL PEARSON, F.R.S.

No. II

On the Construction of Tables and on Interpolation.
Part I. Uni-variate Tables

BY KARL PEARSON, F.R.S.

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TRACTS FOR COMPUTERS

No. II

PREFATORY NOTE

To Nos. II and III

These tracts, Nos. II and III, do not profess to be a complete treatise on the construction of mathematical tables, still less a full mathematical treatment of interpolation. They put together some of the practical processes, which have been found of service in the Biometric Laboratory and state some of the difficulties which have arisen in very heavy recent computations and I would draw the attention of the pure mathematician to the necessity for their solution. As far as I am aware, but I have not made a wide search of the literature, the bi-variate central difference formulae provided are novel. They are those which naturally arise, however, when we come to deal with tables of double entry in practice.

The main doctrine insisted on is that in ordinary mathematical tables accuracy would be gained if the tabulation of first differences were replaced by the tabulation of first central differences, and that in bi-variate tables the tabulation of the two first central differences of both variates is in the bulk of cases the sole method by which the material can be reduced within the bounds of possible publication.

I have received most valuable aid in calculation and in algebraic work from my colleagues Miss Ethel M. Elderton, Miss M. Seegar, and Mr H. E. Soper, and in the preparation of diagrams from Miss A. G. Davin, for which I feel deeply indebted.

KARL PEARSON.

CONTENTS

* The student is urged to practise himself in finding logarithms on this compressed logarithmic table, using a small arithmometer, and checking his results by any non-compressed table to 7 figures.

ON THE CONSTRUCTION OF TABLES AND ON INTERPOLATION

PART I. UNI-VARIATE TABLES

BY KARL PEARSON, F.R.S.

A. *Introductory.*

There are two chief functions of interpolation.

(i) To expedite the construction of new tables.

(ii) To assist the entry into an already constructed table.

Of these functions the former is by far the more important, and this, for reasons which will shortly appear, is also the more arduous.

Given a function of any number of variates this function is said to be tabled when for any number of values of the variates, the numerical value of the function is computed and recorded. If the function be determined by a single variate, the table is termed a *uni-variate* table; if there are several variates required to determine the function the table is termed a *multi-variate* table. As a rule the values of the variates for which the function is calculated proceed by equal intervals, although these equal intervals need by no means be the same throughout the whole range of the variate for which the function is computed*. Again some variates may have large and others small tabular intervals according to the sensitiveness of the function to changes in these variates. The choice of intervals in the variates depends upon a number of considerations which can only be adequately dealt with when the nature of the function and the purposes of the table have been defined. Some of these considerations are antagonistic and a final system of variate-, or as they are frequently termed argument-intervals can only be selected by way of compromise.

* Thus it may be desirable to give a trigonometrical function for every second of angle at one part of a table, while every minute will suffice at another part.

Thus:

(i) The argument-interval should be such that when the table is constructed, the work of entering it for non-tabled values of the variates shall be as expeditious and simple as possible.

With modern computing machines probably the most expeditious method of entering a table is to take definite proportions of the adjacent or neighbouring entries. The ideal method of entering a uni-variate table is to use only the two adjacent entries, i.e. to use only a single "difference." The ideal method of entering a bi-variate table is to use only the four (or, as the method adopted may require, five) adjacent entries, i.e. to use again only first "differences."

(ii) The argument intervals should be such that when the table as planned is completed, the size of it is not such that printing it is prohibitive. Not only may the size render publication impossible, but the labour of computing may be excessive.

Considerations (i) and (ii) are largely antagonistic. The reader has only to consider such a table as that of common logarithms, to 7 figures say, and suppose it were desirable to have a table of $\log_x y$, where the base x varied as well as the argument y. Then, if the variate x were given only 100 values, we should require something like 30,000 octavo pages of figures for the table. In order to fulfil (ii) we should have to take the values of the argument y at far wider intervals than is usual in logarithm tables of single entry. But in doing this we should have to drop the ancient prejudice that in a good table we need first differences only.

In modern practice multi-variate tables are coming more and more into prominence, and what has been suggested in the previous paragraph in the case of two variates is much strengthened when we come to three or more. There is very little hope that in the case of such tables any first difference method of entry will be adequate. A knowledge therefore of the methods of entering such tables becomes essential, and it ought to be familiarised at an early or school stage of practical mathematics. Take cases of tables which it has fallen to the lot of the Department of Applied Statistics to compute during the recent war years. High angled fire is now a commonplace of gunnery, but the height and angle of sight of the objective demand at once a table of double entry. Or again for bombing airplane sights we require both height and speed of airplane. Perhaps the most intricate multi-variate table that we were asked to prepare was a table of the two coordinates in the plane of the rear-sight of the sighting-point for three separate machine guns

(Lewis, Hotchkiss and Vickers) for the four arguments, height, angle of sight, speed, and direction of motion of low-flying airplanes with which the Germans were making the British trenches "unhealthy" towards the end of 1917. Because there were two coordinates, three guns and four variates this table contained a tremendous mass of figures. But except for the values of the arguments tabled, only a rough approximation could be reached if first difference interpolation was applied.

With the wide practical use of multi-variate tables—one instance of which, that of ballistics, only has been cited—it becomes needful to indicate the difficulties of multi-variate interpolation formulae, and suggest how they can be met. It is usual for the computer to associate every table entry or every pair of tabled entries with a single difference. He may be acquainted with some tables which provide two.

When we come to multi-variate tables the number of differences corresponding to a single table entry, if we have to use fairly considerable argument ranges may be considerably increased. Thus with a table of double entry only we may require to associate each entry with perhaps four or five differences. Should we interpolate by "forward" differences, we need at least double this number! Hence it is easy to grasp that extension of argument intervals while reducing the number of entries to manageable size, may force us again to increase our table to unwieldy bulk by the necessity for including a large series of differences. It is quite possible for the differences even in a bi-variate table to occupy as much or even twice as much space as the tabled values of the function.

We thus reach our third consideration, namely:

(iii) The argument intervals must not be so large that the space thus gained is not once more largely lost by the necessity for tabling a very large number of differences. We cannot expect if a table is to be of utility, that the computer should be left to calculate, say all the fourth differences he needs to enter the table. There must be associated tables of differences provided. The problem will always be how to present these differences, and how to keep them in bounds in any given case. The very fact that the argument intervals are large will mean that the differences themselves are large and occupy considerable space.

The reader will see that our considerations (i), (ii) and (iii) are in great part antagonistic, and that it requires much judgment in the case of a given table to get a minimum of size combined with a maximum of utility to the user.

Reverting to the usual method of constructing a table, which for the present we will suppose uni-variate, the start is made by constructing a "frame" from the formula, series or integral providing the function by straightforward calculation, which may mean use of quadratures*. The entries involved in this frame will usually be to two more decimal places than will be retained in the final table. As a rule with modern computing machines it is more rapid and efficient to interpolate between the entries of this frame by Lagrangian formulae using the 4, 6, 8 or 10 adjacent entries as the case may require, rather than by constructing an elaborate system of differences for the frame, which at any rate will have to be discarded as of no value for the completed table, and will very probably have to be discarded still earlier, if it turns out that the "panels" of the frame have been selected at too great intervals for effective interpolation. Of course at first start the panel-intervals are selected largely by appreciation; their adequacy must then be tested. Treating the panel interval as unit our frame entries from the middle of the panel we desire to fill in will be

$$\dots y_{-4\cdot5},\ y_{-3\cdot5},\ y_{-2\cdot5},\ y_{-1\cdot5},\ y_{-\cdot5},\ y_{+\cdot5},\ y_{1\cdot5},\ y_{2\cdot5},\ y_{3\cdot5},\ y_{4\cdot5}\ \dots$$

We now desire the interpolation of, it may be, four values, $y_{-\cdot3}$, $y_{-\cdot1}$, $y_{+\cdot1}$, $y_{+\cdot3}$ between $y_{-\cdot5}$ and $y_{+\cdot5}$ or again of nine values

$$y_{-\cdot4},\ y_{-\cdot3},\ y_{-\cdot2},\ y_{-\cdot1},\ y_0,\ y_{+\cdot1},\ y_{+\cdot2},\ y_{+\cdot3},\ y_{+\cdot4}$$

in the same panel. Any value such as $y_{-\cdot3}$ will be provided by a formula of the type

$$y_{-\cdot3} = \dots\ {}_{-\cdot3}n_{-4\cdot5}\,y_{-4\cdot5} + {}_{-\cdot3}n_{-3\cdot5}\,y_{-3\cdot5} + {}_{-\cdot3}n_{-2\cdot5}\,y_{-2\cdot5} + {}_{-\cdot3}n_{-1\cdot5}\,y_{-1\cdot5} + {}_{-\cdot3}n_{-\cdot5}\,y_{-\cdot5}$$
$$+ {}_{-\cdot3}n_{+\cdot5}\,y_{+\cdot5} + {}_{-\cdot3}n_{+1\cdot5}\,y_{+1\cdot5} + {}_{-\cdot3}n_{+2\cdot5}\,y_{+2\cdot5} + {}_{-\cdot3}n_{+3\cdot5}\,y_{+3\cdot5} + {}_{-\cdot3}n_{+4\cdot5}\,y_{+4\cdot5} + \dots$$

the y's being the frame values and the n's tabled arithmetical values always at hand in the computing laboratory.

We shall start, say, by using a six-entry interpolation formula, then we shall apply an eight-entry formula. If both give the *same* result we may be fairly satisfied that our panel intervals are small enough. If not we proceed to consider a ten-entry formula and see if it agrees with the eight-entry interpolation. Beyond this it is probably not advisable to proceed, but the ten-entry interpolation may be checked against directed calculation of the interpolates.

In the Department of Applied Statistics eight- and ten-entry interpolation formulae have been in constant use for wide areas of large tables. They repre-

* Discussion of quadrature methods will be provided in a later tract of this series.

sent one continuous process on the machine* and have not been found too lengthy for practical use. Experience seems to show, however, that where these fail, it is more economical to reduce the frame panels than to use higher entry interpolation formulae.

Throughout the process: (i) interpolated values must be checked by occasional directly calculated values, (ii) the system of differences should be taken; the new differences obtained by each interpolated value as it is entered, being a very desirable, if not wholly adequate test of the accuracy of the machinings, and further supplying the system of differences which will eventually be published in whole or part with the finished table itself.

It is a very wise rule to carry on the difference system simultaneously with the entry system. The differences are not only a check on the accuracy of the work in progress, but are further an indication of whether the panel intervals are too large, or unnecessarily small, and again of whether the intervals of the final table will be too large for practical interpolation, or on the other hand too small, i.e. whether we are wasting space, which is so essential for the proper presentation of multi-variate tables.

This leads us to a remark which may be of service to the computer. It is by no means needful either in the panel intervals of the frame, or in the entry intervals of the final table to maintain an equality of interval. On the contrary having regard to the economy of labour and the economy of space, changes of interval in both frame and final table—especially in the case of multi-variate tables—are to be commended. Provision in both cases must, however, be made either by "overlapping" of the areas or by "bridging formulae" to meet the interpolation difficulties both in the frame filling-in and in the practical use of the table for interpolating values at the boundaries of changes of interval.

* The machine used should be one that carries at least three figures on the slide and if possible carries throughout the whole range. The old-fashioned machines carrying one figure only are sure to give trouble, because subtractions and additions will alternate, and the labour is much increased if separate products have to be written down. Further the slide which carries throughout enables us to work out the continuous process by what in the old school arithmetics is spoken of as contracted multiplication. For example if we require a final table to eight figures, we may use our y's and n's each to ten figures, but work back on the slide using the left-hand figure of the multiplier first, and ceasing to multiply by that figure when we reach what will be the 11th or 12th figure in the final result. The ultimate value of the continued operation on the slide will be certainly true to the 9th or 10th figure and therefore correct to the required 8th figure. We have found for such purposes the 12×12 recording to 24 figures new type Trinks-Brunsviga with automatic carriage a very satisfactory machine.

The reader will have observed that in our discussion above of the process of filling in panel intervals we have indicated the use of what may be termed mid-panel interpolation formulae; we have interpolated between $y_{-\cdot 5}$ and $y_{+\cdot 5}$ from the same number of frame entries on either side of this mid-panel interval. Although this is not involved in the Lagrangian formula, experience will rapidly show the computer its great desirability. If we interpolate in side-panels the results will be found to grow more and more unsatisfactory as we approach the end panel on either side. So unsatisfactory will they become that it may be necessary to reduce the panel interval, and no increase in the number of *one-sided* interpolants will often avail to better matters. This merely amounts to saying in more technical language that "forward differences" even if very large in number will not necessarily give satisfactory results. In common sense language we want the adequate number of "nearest points" to get the best result, not a very large number of more and more distant points. In our own mind there is little doubt that "forward difference" formulae are chiefly of theoretical interest; that modern methods of computing demand mid-interval interpolation formulae of the Lagrangian type, or what corresponds to them central difference formulae. The great advantage of central difference formulae, especially in multi-variate tables, consists in the fact that we do not use a large number of differences for a single entry, but a few central differences of each of a number of neighbouring entries. In this way exactitude is combined with economy of space.

At the borders of a table, however, other considerations come into play. It will be impossible to use without adaptations mid-panel interval interpolations; we cannot get our central differences for boundary entries, if no values beyond the boundary are known or, perhaps, indeed conceivable. At such table borders we may try our luck with side- or end-panel interpolation—i.e. with forward differences—but we may be in more senses than one on the edge of a precipice*. The computer may try many things, but he will probably find it most satisfactory and most economical of time to calculate out by brute force the entries he requires, until he finds side-panel interpolation giving the same results as his computed values, then he may continue to use it, till he comes to his first mid-panel interpolation, which he will then by shifting retain until he comes to another boundary or a change of interval. In some cases the function while having no physical meaning, or perhaps

* The function may be finite at the boundary and imaginary beyond; it may become asymptotic or there may be a cuspidal ridge; mathematical difficulties which defeat the use of central differences, but which experience soon shows are usually wholly beyond treatment by forward differences.

having no value necessary for tabulation*, can be extended mathematically beyond the table value and thus mid-panel interpolation can be carried right up to the boundary of the table. Such cases are very important when we are tabulating central differences at a table boundary, for it will be clear that without values outside the boundary we cannot use that method of interpolation and must trust at the boundary to forward difference methods. Yet it is precisely in those cases where the mathematical function ceases or becomes imaginary on the other side of the boundary, that forward differences are likely to give bad results. Occasionally it is feasible to supplement the fundamental table by an auxiliary table at the boundary, which will help us over at least some of our difficulties with, perhaps, one or more of the variates.

As illustration consider the Incomplete Γ-function, i.e.

$$I(u, p) = \frac{\int_0^{u\sqrt{p+1}} e^{-v} v^p \, dv}{\Gamma(p+1)}$$

$$= \frac{(p+1)^{\frac{1}{2}(p-1)}}{\Gamma(p+1)} u^{p+1} e^{-u\sqrt{p+1}} \left(1 + \frac{u\sqrt{p+1}}{p+2} + \frac{u^2(p+1)}{(p+2)(p+3)} + \frac{u^3(p+1)^{\frac{3}{2}}}{(p+2)(p+3)(p+4)} + \ldots\right).$$

Clearly the expression becomes imaginary when p being fractional u is negative, or again when p is negative and greater than unity.

Taking logarithms to base 10, we have:

$$\log I(u, p) = \tfrac{1}{2}(p-1)\log(p+1) - \log\Gamma(p+1) - \sqrt{p+1}\, u \log e + (p+1)\log u$$
$$+ \log\left(1 + \frac{u\sqrt{p+1}}{p+2} + \frac{u^2(p+1)}{(p+2)(p+3)} + \frac{u^3(p+1)^{\frac{3}{2}}}{(p+2)(p+3)(p+4)} + \ldots\right).$$

Now the imaginary term introduced by negative u is seen to lie in $(p+1)\log u$. Let us call $I'(u, p)$ the function $I(u,p)/u^{p+1}$, then

$$\log I'(u, p) = \log I'(0, p) - u\sqrt{p+1}\log e + \log E(u, p),$$

where $E(u, p)$ is the series above. This series for small values of u, say not

* For example in range tables we do not tabulate forward and rearward ranges, but for many purposes the fact that we can mathematically proceed through vertical fire to rearward fire is most helpful for interpolation. Indeed to hit a rapidly oncoming low flying airplane at a high angle of sight it might be needful, were it practicable, to fire over the shoulder, and it may be desirable to calculate one or more such values for interpolation purposes.

above 1·7 admits of easy computation, and it may be applied for u both positive and negative. We can readily deduce, however, that:

$$\log I'(-u, p) = 2\log I'(0, p) - \log I'(u, p)$$

$$+\log\left(1 + \frac{u^2}{(p+3)}\frac{(p+1)^2}{(p+2)^2} + \frac{u^4(p+1)^3}{(p+2)(p+3)^2(p+4)(p+5)}\right.$$

$$+\frac{u^6(p+1)^4}{(p+2)(p+3)(p+4)^2(p+5)(p+6)(p+7)}$$

$$\left.+\frac{u^8(p+1)^5}{(p+2)(p+3)(p+4)(p+5)^2(p+6)(p+7)(p+8)(p+9)}+\dots\right),$$

this series converging with great rapidity.

Thus $\log I'(-\cdot 1, p)$ and $\log I'(-\cdot 2, p)$ can be rapidly found from $\log I'(\cdot 1, p)$ and $\log I'(\cdot 2, p)$ and the central differences of 2nd and 4th order found for $\log I'(0, p)$, i.e. at the start or boundary of the table for the u variate. It will be further found that for $p = 0$ to 10, $u = 0$ to $1\cdot 5$ the differences of the function $\log I'(u, p)$ are far more convergent than those of $I(u, p)$, and for exact work the auxiliary table within these boundaries is more effective than the fundamental table. Of course it involves adding $(p+1)\log u$ to $\log I'(u, p)$ and then looking out the antilog to find $I(u, p)$. Within the above limits of p and u we should probably have to lessen our table intervals to a tenth, or increase this portion of the table ten-fold to obtain a corresponding accuracy for $I(u, p)$.

Such an illustration indicates the double purpose of finding central differences at a boundary by "extrafinial" values; and also the economy of space near a boundary which may arise from the use of an auxiliary function table.

Unfortunately neither of these processes will apply to the "function-ridge" at $p = -1$. Nothing but the calculation of the function for very small tabled intervals of p between at least $p = -1$ and $p = -\cdot 5$ will suffice to determine the Incomplete Γ-function in this region. We must then use central difference methods for the u and forward difference methods (or those of p. 13) for the p interpolation. It is especially in the case of multi-variate tables that interpolation both for frame and for the actually completed table becomes a matter very often of considerable difficulty. No standardised method of procedure can be adopted and the computer preparing a new table can only be warned that he must adapt his methods to his function, and that

where no extrafinial values can be computed the determination of proximo-finial values and differences may prove by far the hardest part of the task*.

Part I. *Uni-variate Interpolation.*

The reader is supposed to be acquainted with the simple operators of the Calculus of Finite Differences. Namely, if z_x be a function of the variate x

$$Ez_r = z_{r+1},$$

$$E^x z_r = z_{r+x}, \quad E^x z_0 = z_x, \text{ etc.}$$

$$\Delta z_r = z_{r+1} - z_r$$

or,

$$1 + \Delta = E.$$

Accordingly $E^x = (1+\Delta)^x$, and

$$z_x = E^x z_0 = (1+\Delta)^x z_0$$

$$= z_0 + \frac{x}{1\,!}\Delta z_0 + \frac{x(x-1)}{2\,!}\Delta^2 z_0 + \frac{x(x-1)(x-2)}{3\,!}\Delta^3 z_0 + \dots \qquad \dots\text{(i)}.$$

This is the fundamental "forward difference" formula, $\Delta^r z_0$ being the rth forward difference of z_0. Hence if we wished to work with differences of z_0 only we should have to table four forward differences for each entry if we wanted to go to fourth forward difference accuracy—often in itself not a very high degree—and to n differences if we wanted nth difference accuracy. The reader should keep the accompanying scheme in his mind.

$$\begin{array}{llllll}
 & \Delta & \Delta^2 & \Delta^3 & \Delta^4 & \Delta^5 \\
z_0 & & & & & \\
 & z_1 - z_0 & & & & \\
z_1 & & z_2 - 2z_1 + z_0 & & & \\
 & z_2 - z_1 & & z_3 - 3z_2 + 3z_1 - z_0 & & \\
z_2 & & z_3 - 2z_2 + z_1 & & z_4 - 4z_3 + 6z_2 - 4z_1 + z_0 & \\
 & z_3 - z_2 & & z_4 - 3z_3 + 3z_2 - z_1 & & z_5 - 5z_4 + 10z_3 - 10z_2 + 5z_1 - z_0, \\
z_3 & & z_4 - 2z_3 + z_2 & & z_5 - 4z_4 + 6z_3 - 4z_2 + z_1 & \\
 & z_4 - z_3 & & z_5 - 3z_4 + 3z_3 - z_2 & & \\
z_4 & & z_5 - 2z_4 + z_3 & & & \\
 & z_5 - z_4 & & & & \\
z_5 & & & & &
\end{array}$$

* Another method of computing *central* differences at the border of a table may be of value. Namely to consider $\delta^2 u_0$ unknown and determine its value from a directly computed interpolate. If $\delta^4 u_0$ be not insensible then two directly computed interpolates will give two equations to find these unknowns. For example in a table of antilogs we have

$$\begin{array}{lll} 00 & 10000000 & \delta^2 u_0 = ? \\ 01 & 10232930 & \delta^2 u_1 = 5425, \end{array}$$

while $\delta^4 u_0$ and $\delta^4 u_1$ are negligible.

Compute the antilog of 005, i.e. 10115795. Hence

$$10115795 = \tfrac{1}{2}(1 + 10232930) - \tfrac{1}{24}\, 1{\cdot}5\, (\delta^2 u_0 + 5425),$$

hence we have:

$$670 = \tfrac{1}{16}(\delta^2 u_0 + 5425)$$

$$\delta^2 u_0 = 5295.$$

When the z's are numbers the differencing can, even if the values are to ten figures, be done rapidly on a machine. The subtrahend of the previous subtraction is not cleared with the slide, but transferred to the slide and the new subtrahend then takes its place. The work is tested by simply adding up any column; the successive numbers should be those in the previous column. This is best done on a recording adding machine, examining the totals at frequent intervals, if we are working columns of 50 to 100 differences.

In the above scheme the first entries in each column are the successive forward differences of z_0; the second entries are those of z_1 and so on. The entries on the same line as any function-value are the *central* differences of that value. These we shall represent by δ^2, δ^4, etc.

Thus we have:

$$\delta^2 z_1 = z_2 - 2z_1 + z_0 = \Delta^2 z_0 \quad \text{......(ii)}.$$

$$\delta^2 z_2 = z_3 - 2z_2 + z_1 = \Delta^2 z_1 \quad \text{......(iii)}.$$

$$\delta^4 z_2 = z_4 - 4z_3 + 6z_2 - 4z_1 + z_0 = \Delta^4 z_0 \quad \text{......(iv)}.$$

Again we have:

$$\Delta^3 z_0 = \delta^2 z_2 - \delta^2 z_1 \quad \text{......(v)},$$

$$\Delta^5 z_0 = \delta^4 z_3 - \delta^4 z_2 \quad \text{......(vi)}.$$

Clearly unless we can find function values like z_{-1} and z_{-2} one cannot obtain the first or second (δ^2 or δ^4) central differences of z_0 or the second (δ^4) central difference of z_1.

Now the above values indicate that all odd differences can be expressed in terms of lower, even central, differences. Also we see that

$$\left.\begin{aligned} \delta^2 z_2 &= z_3 + z_1 - 2z_2 \\ \delta^4 z_2 &= \delta^2 z_3 + \delta^2 z_1 - 2\delta^2 z_2 \\ \delta^6 z_2 &= \delta^4 z_3 + d^4 z_1 - 2\delta^4 z_2 \end{aligned}\right\} \quad \text{......(vii)},$$

and so on.

Thus the requisite central differences can be calculated at once without introducing the odd differences columns at all, by the simple process of adding from the previous column the entry in the row above to the entry in the row below and subtracting twice the entry on the same row. Thus:

	δ^2	δ^4
z_0		
z_1	$z_2+z_0-2z_1$	
z_2	$z_3+z_1-2z_2$	$(z_4+z_2-2z_3)+(z_2+z_0-2z_1)-2\,(z_3+z_1-2z_2)$
z_3	$z_4+z_2-2z_3$	$(z_5+z_3-2z_4)+(z_3+z_1-2z_2)-2\,(z_4+z_2-2z_3)$
z_4	$z_5+z_3-2z_4$	
z_5		

The process on the machine is practically very rapid and half the labour of differencing is saved.

We will now convert our forward difference formulae into an "end-panel" or "first-panel" central difference formula. If we use only δ^2 and δ^4 we shall be as accurate as if we had worked with a fifth difference formula.

Substituting we have, neglecting sixth differences:

$$z_x = z_0 + x(z_1 - z_0) + \frac{x(x-1)}{2!}\delta^2 z_1 + \frac{x(x-1)(x-2)}{3!}(\delta^2 z_2 - \delta^2 z_1)$$

$$+ \frac{x(x-1)(x-2)(x-3)}{4!}\delta^4 z_2 + \frac{x(x-1)(x-2)(x-3)(x-4)}{5!}(\delta^4 z_3 - \delta^4 z_2) \qquad \text{...........(viii).}$$

Write $x = \theta$, $1 - x = \phi$, and we obtain for a

First-panel Central Difference Formula:

$$\begin{aligned} z_\theta = {} & \phi z_0 + \theta z_1 - \tfrac{1}{6}\theta\phi\,\{(4+\phi)\,\delta^2 z_1 - (1+\phi)\,\delta^2 z_2\} \\ & - \tfrac{1}{120}\theta\phi\,(1+\phi)(2+\phi)\,\{(8+\phi)\,\delta^4 z_2 - (3+\phi)\,\delta^4 z_3\} \\ & - \tfrac{1}{5040}\theta\phi\,(1+\phi)(2+\phi)(3+\phi)(4+\phi)\,\{(12+\phi)\,\delta^6 z_3 - (5+\phi)\,\delta^6 z_4\} \\ & + \ldots \qquad \text{..(viii)}^{\text{bis}}. \end{aligned}$$

This formula enables us to interpolate on the border of a table from the central differences, which can be found from the entries of that table, i.e. without extrafinial values being calculated. If the sixth difference term be negligible, then the tabulation of δ^2 and δ^4 for the actual entries, where available, will be adequate, for no central difference of z_0 and only one of z_1 is required.

We now pass to the panel between z_1 and z_2 and suppose the interpolate to divide this as θ' to ϕ'. We have at once $\theta = 1 + \theta'$, $\phi = \phi' - 1$, and remembering that we can write $z_0 = \delta^2 z_1 - z_2 + 2z_1$, we find after transforming to θ' and ϕ' the

Second-panel Central Difference Formula:

$$\begin{aligned} z_{\theta'} = {} & \phi' z_1 + \theta' z_2 - \tfrac{1}{6}\theta'\phi'\,\{(1+\phi')\,\delta^2 z_1 + (1+\theta')\,\delta^2 z_2\} \\ & + \tfrac{1}{120}\theta'\phi'\,(1+\theta')(1+\phi')\,\{(7+\phi')\,\delta^4 z_2 - (2+\phi')\,\delta^4 z_3\} \\ & + \tfrac{1}{5040}\theta'\phi'\,(1+\theta')(1+\phi')(2+\phi')(3+\phi')\,\{(11+\phi')\,\delta^6 z_3 - (4+\phi')\,\delta^6 z_4) \\ & - \ldots\ldots \qquad \text{..(ix).} \end{aligned}$$

We will now proceed to the third panel between z_2 and z_3, and write $\theta'' = \theta'$, $\phi'' = 1 + \phi'$. We must get rid of z_1 and $\delta^2 z_1$ by the formulae:

$$z_1 = \delta^2 z_2 - z_3 + 2z_2, \quad \delta^2 z_1 = \delta^4 z_2 - \delta^2 z_3 + 2\delta^2 z_2,$$

and we find after some reductions the:

Third-panel Central Difference Formula:

$$\begin{aligned} z_{\theta''} = {} & \phi'' z_2 + \theta'' z_3 - \tfrac{1}{6}\theta''\phi'' \{(1+\phi'')\,\delta^2 z_2 + (1+\theta'')\,\delta^2 z_3\} \\ & + \tfrac{1}{120}\,\theta''\phi''(1+\theta'')(1+\phi'')\{(2+\phi'')\,\delta^4 z_2 + (2+\theta'')\,\delta^4 z_3\} \\ & - \tfrac{1}{5040}\,\theta''\phi''(1+\theta'')(1+\phi'')(2+\theta'')(2+\phi'')\{(10+\phi'')\delta^6 z_3 - (3+\phi'')\,\delta^6 z_4\} \\ & + \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(x)}. \end{aligned}$$

The law of development is now clear and we can write down the general result on the supposition that each term has its full complement of central differences. Namely if the panel be between z_s and z_{s+1} with the intervals θ and ϕ

$$\begin{aligned} z_\theta = {} & \phi z_s + \theta z_{s+1} - \tfrac{1}{6}\theta\phi\{(1+\phi)\,\delta^2 z_s + (1+\theta)\,\delta^2 z_{s+1}\} \\ & + \tfrac{1}{120}\,\theta\phi(1+\theta)(1+\phi)\{(2+\phi)\,\delta^4 z_s + (2+\theta)\,\delta^4 z_{s+1}\} \\ & - \tfrac{1}{5040}\,\theta\phi(1+\theta)(1+\phi)(2+\theta)(2+\phi)\{(3+\phi)\,\delta^6 z_s + (3+\theta)\,\delta^6 z_{s+1}\} \\ & + \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xi)}. \end{aligned}$$

This is Everett's Central Difference Formula. It is in every respect superior to the Forward Difference Formula. It fails, however, in special cases when we come to proximo-finial regions. For such regions the First, Second, and Third-panel Central Difference Formulae given above have been introduced. A further slight objection to the formula is that it ultimately involves a knowledge of an *even* number of table entries. For example, if we go to δ^6 the differences are based on a knowledge of

$$z_{s-3},\ z_{s-2},\ z_{s-1},\ z_s,\ z_{s+1},\ z_{s+2},\ z_{s+3},\ z_{s+4}.$$

When it is desirable to use an odd number of table entries Everett's formula cannot be used.

While not advantageous for interpolating a frame, it is excellent for entering an already constructed table, supplied with the values of δ^2 and δ^4. If it is needful to go to δ^6 then we can easily find sixth central differences from

$$\delta^6 z_s = \delta^4 z_{s-1} + \delta^4 z_{s+1} - 2\delta^4 z_s.$$

The use of Everett's formula will be much expedited by the publication of tables of the coefficients, which are now being computed. Assuming, that

as usual, our differences correspond to the last figures of the tabled entries, then as a guide to what differences need be used, we remark that it is unnecessary to use any second difference (δ^2) under 4, any fourth difference (δ^4) under 20 and any sixth difference (δ^6) under 100.

If only second central differences are attached to a table we can express our formula in terms of these only. Thus up to and including terms of the fifth difference order we have:

$$\begin{aligned} z_\theta = \phi z_s + \theta z_{s+1} &+ \tfrac{1}{120}\,\theta\phi\,(1+\theta)\,(1+\phi)\,\{(2+\phi)\,\delta^2 z_{s-1} + (2+\theta)\,\delta^2 z_{s+2}\} \\ &- \tfrac{1}{120}\,\theta\phi\,(1+\phi)\,(2+\phi)\,(3\theta+8)\,\delta^2 z_s \\ &- \tfrac{1}{120}\,\theta\phi\,(1+\theta)\,(2+\theta)\,(3\phi+8)\,\delta^2 z_{s+1} \quad \text{...............(xii).} \end{aligned}$$

Similarly it is possible to write down a central difference formula correct up to and including the seventh order differences which will involve only six second differences.

It is clear, however, that as we have also to use z_s and z_{s+1} we should not gain much in practical application—given machine computing—on a Lagrangian formula of the type:

$$\begin{aligned} z_\theta = n_{s-3}z_{s-3} + n_{s-2}z_{s-2} + n_{s-1}z_{s-1} + n_s z_s + n_{s+1}z_{s+1} + n_{s+2}z_{s+2} \\ + n_{s+3}z_{s+3} + n_{s+4}z_{s+4}, \end{aligned}$$

for we should have a continuous process of multiplying in pairs the n's and z's and adding the products. Relative speed of working would largely depend on whether the (θ, ϕ) coefficients or the n's were already calculated and tabled.

The n's will not probably be known *a priori* for any special individual value of z likely to be demanded from a completed table. On the other hand they will be calculated when it is our business to construct a table from its frame, because the same values of the n's will be used over and over again for each panel. Thus our experience has shown us that for entering a completed table central difference formulae are most valuable, but for constructing a table formulae of the Lagrangian type are the best.

We now turn to such formulae. We shall go a roundabout way to deduce them because it is desirable to indicate that forward difference, central difference and Lagrangian formulae are all the same thing in reality, only expressed in different forms, and that accordingly one general criticism applies to them all, which we shall discuss later.

Returning to (i) let us insert the z-values of the differences as given in the scheme on p. 11 and rearrange under separate z's. We find:

$$z_x = z_0\left(1 - x + \frac{x(x-1)}{2\,!} - \frac{x(x-1)(x-2)}{3\,!} + \dots\right)$$

$$+ z_1 x\left(1 - (x-1) + \frac{(x-1)(x-2)}{2\,!} - \frac{(x-1)(x-2)(x-3)}{3\,!} + \dots\right)$$

$$+ z_2\frac{x(x-1)}{1\,.\,2}\left(1 - (x-2) + \frac{(x-2)(x-3)}{2\,!} - \frac{(x-2)(x-3)(x-4)}{3\,!} + \dots\right)$$

$$+ z_3\frac{x(x-1)(x-2)}{3\,!}\left(1 - (x-3) + \frac{(x-3)(x-4)}{2\,!} - \frac{(x-3)(x-4)(x-5)}{3\,!} + \dots\right)$$

$$+ \text{etc.} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(xiii)}.$$

Now it is easy to show that

$$1 - (x-r) + \frac{(x-r)(x-r-1)}{2\,!} - \frac{(x-r)(x-r-1)(x-r-2)}{3\,!} + \dots$$

is equal to unity for $x = r$ and vanishes for $x = r+1,\ r+2, \dots n-1$, if there be n values of z, or we proceed to the $(n-1)$th order of differences. Hence this expression must be of the form

$$C_r(x-r-1)(x-r-2)\dots\dots\dots(x-n+1).$$

But when $x = r$ the value of the expression is unity. Accordingly we have:

$$C_r(-1)(-2)\dots\dots\dots(-n+r+1) = 1,$$

or
$$C_r = \frac{(-1)^{n-r-1}}{(n-r-1)\,!}.$$

Thus the coefficient of z_r in z_x is seen to be:

$$\frac{x(x-1)(x-2)\dots(x-r+1)(x-r-1)\dots(x-n+1)}{r(r-1)(r-1)\dots\dots\dots\dots\dots\dots\dots\dots(r-n+1)},$$

an expression equal to unity at $x = r$ and vanishing at

$$x = 0,\ 1,\ 2 \dots r-1,\ r+1, \dots n-1.$$

Accordingly the forward difference formula up to the $(n-1)$th order of differences is seen to be a parabola of the $(n-1)$th order:

$$z_x = c_0 + c_1 x + c_2 x + \dots + c_{n-1}x^{n-1}$$

with the constants $c_0, c_1, \dots c_{n-1}$ so chosen that this parabola passes through the n points where the ordinates are $z_0,\ z_1,\ z_2 \dots z_{n-1}$. We can at once generalise the expression thus reached from the forward difference formula

and make the abscissae of the points $a_0, a_1, a_2, \dots a_{n-1}$ instead of $0, 1, 2, \dots n-1$, thus reaching the general Lagrangian formula in its most useful form:

$$z_x = z_0 \frac{(x-a_1)(x-a_2)(x-a_3)\dots(x-a_{n-1})}{(a_0-a_1)(a_0-a_2)(a_0-a_3)(a_0-a_{n-1})}$$
$$+ \frac{z_1(x-a_0)(x-a_2)(x-a_3)\dots(x-a_{n-1})}{(a_1-a_0)(a_1-a_2)(a_1-a_3)\dots(a_1-a_{n-1})}$$
$$+ z_2 \frac{(x-a_0)(x-a_1)(x-a_3)\dots(x-a_{n-1})}{(a_2-a_0)(a_2-a_1)(a_2-a_3)\dots(a_2-a_{n-1})} + \dots$$
$$+ z_{n-1} \frac{(x-a_0)(x-a_1)(x-a_3)\dots(x-a_{n-2})}{(a_{n-1}-a_0)(a_{n-1}-a_1)(a_{n-1}-a_3)\dots(a_{n-1}-a_{n-2})} \quad \dots\text{(xiv)}.$$

This formula would be most cumbersome, if we had to insert a given value of x for a single interpolation into a table, but as we shall shortly proceed to show it is very valuable for computing interpolates at regular intervals in the panels of a frame.

We have now shown that forward difference formulae, central difference formulae and Lagrangian formulae, when the interpolants are spaced equally apart are really different aspects of the same process, i.e. running a parabola of the $(n-1)$th order through n points. The assumption made therefore in all these interpolation methods is at bottom the same, namely that by taking a sufficiently high order parabola we can approximate closely to the true function. But the differential coefficients of such a parabola are at every point finite and continuous. To get therefore a good correspondence of the function and the parabola the above conditions should hold for the function. The problem is like that of legitimacy of expansion by Maclaurin's Theorem. As a rule the interpolation formulae work, but once in a while bitter experience forces us up against cases in which increasing the number of differences, or the number of points through which the Lagrangian parabola is passed, is quite ineffectual as a method of obtaining accurate interpolates. In one case effective interpolation over very wide panels can be carried out with a quite low difference order formula; in another, with relatively small panels and high differences, interpolations appear very unsatisfactory. The reader has only to try his hand at interpolating between 1 and 2 in an ordinary table of square or cube roots to appreciate this! Suppose he wanted to determine the cube root of 1·1067; he would not use high differences and interpolate between 1 and 2, but he would interpolate between 1106 and 1107, i.e. reduce his interpolant interval to practically $\frac{1}{1000}$th of its

value between 1 and 2*. This reduction of interval is possible in a table of cubes, but in most tables it is not feasible, and we are up against the difficulty of making our intervals so small that a high order parabola will adequately represent the function without at the same time exceeding the possibilities of printing. It is in such cases that the help of the mathematician is needed to transform the function, or devise an auxiliary function, from which the non-interpolable factor has been removed.

It will be observed that (xiv) can be applied to either an odd or even number of table entries, but that Everett's Central Difference formula really uses an even number of entries. We accordingly ask whether it is not possible to obtain a central difference formula involving an odd number of points. To reach this end we do not interpolate into the interval between two entries, but into the nearest space round a point. We shall term Everitt's central difference formula a "mid-panel central difference formula," and that we now proceed to investigate a "mid-point central difference formula." It should be applied only within the interval $x = -\frac{1}{2}$ to $x = \frac{1}{2}$, the origin being taken at the value z_0 of the function, and the entries used being:

$$z_{-s},\ z_{-s+1},\ \dots\ z_{-2},\ z_{-1},\ z_0,\ z_{+1},\ z_{+2},\ \dots\ z_{s-1},\ z_s.$$

Returning to equation (viii), p. 13, it will be remembered that we had to retain differences of z_2, z_3, etc. because we could not use points on the negative side of z_0, i.e. z_{-1}, z_{-2}, etc. This restriction is now removed. We can always replace z_2, z_3, etc. by expressions in terms of z_0, z_1, z_{-1} and their central differences. Thus the following expressions will often be found useful:

$$\begin{aligned}
z_2 &= \delta^2 z_1 + 2z_1 - z_0,\\
z_3 &= \delta^4 z_1 + 4\delta^2 z_1 + 3z_1 - \delta^2 z_0 - 2z_0,\\
z_4 &= \delta^6 z_1 + 6\delta^4 z_1 + 10\delta^2 z_1 + 4z_1 - \delta^4 z_0 - 4\delta^2 z_0 - 3z_0,\\
z_5 &= \delta^8 z_1 + 8\delta^6 z_1 + 21\delta^4 z_1 + 20\delta^2 z_1 + 5z_1 - \delta^6 z_0 - 6\delta^4 z_0 - 10\delta^2 z_0 - 4z_0 \quad \dots\text{(xv)},
\end{aligned}$$

or in terms of the central differences of z_1 and z_{-1} only:

$$\begin{aligned}
z_2 &= \delta^2 z_1 + 2z_1 - z_0,\\
z_3 &= \delta^4 z_1 + 4\delta^2 z_1 + 2z_1 - z_{-1},\\
z_4 &= \delta^6 z_1 + 6\delta^4 z_1 + 9\delta^2 z_1 - \delta^2 z_{-1} + 2z_1 - 2z_{-1} + z_0,\\
z_5 &= \delta^8 z_1 + 8\delta^6 z_1 + 20\delta^4 z_1 - \delta^4 z_{-1} + 16\delta^2 z_1 - 4\delta^2 z_{-1} + 3z_1 - 2z_{-1} \quad \dots\dots\text{(xv)}^{\text{bis}}.
\end{aligned}$$

The reader can verify these by remembering that

$$\delta^2 = \frac{(E-1)^2}{E}, \text{ or } \delta = \sqrt{E} - \frac{1}{\sqrt{E}}.$$

* This amounts to saying that, except for small stretches of the function, $x^{\frac{1}{3}}$ cannot for small values of x be adequately replaced by a parabolic curve.

From these results by operating with δ^2 we can obtain the central differences of z_2, z_3, z_4, z_5 in terms of those of z_1, z_0 and z_{-1}.

Substituting such values obtained from (xv) in equation (viii) and rearranging in order of differences we deduce:

$$z_x = z_0(1-x) + xz_1 + \frac{x(x^2-1)}{3!}\delta^2 z_1 - \frac{x(x-1)(x-2)}{3!}\delta^2 z_0$$
$$+ \frac{x(x^2-1)(x^2-4)}{5!}\delta^4 z_1 - \frac{x(x^2-1)(x-2)(x-3)}{5!}\delta^4 z_0$$
$$+ \frac{x(x^2-1)(x^2-4)(x^2-9)}{7!}\delta^6 z_1 - \frac{x(x^2-1)(x^2-4)(x-3)(x-4)}{7!}\delta^6 z_0$$
$$+ \text{etc.} \quad \text{.......................(xvi).}$$

This formula as it stands may be of use if we desire to use only central differences on one side of the origin. Writing down a similar expression for the other side we have:

$$z_x = z_0(1+x) - xz_{-1} - \frac{x(x^2-1)}{3!}\delta^2 z_{-1} + \frac{x(x+1)(x+2)}{3!}\delta^2 z_0$$
$$- \frac{x(x^2-1)(x^2-4)}{5!}\delta^4 z_{-1} + \frac{x(x^2-1)(x+2)(x+3)}{5!}\delta^4 z_0$$
$$- \frac{x(x^2-1)(x^2-4)(x^2-9)}{7!}\delta^6 z_{-1} + \frac{x(x^2-1)(x^2-4)(x+3)(x+4)}{7!}\delta^6 z_0$$
$$- \text{etc.} \quad \text{.......................(xvi)}^{\text{bis}}.$$

Taking the mean of (xvi) and $\text{(xvi)}^{\text{bis}}$ we reach:

$$z_x = z_0 + \tfrac{1}{2}x(z_1 - z_{-1}) + \frac{x(x^2-1)}{3!\,2}(\delta^2 z_1 - \delta^2 z_{-1}) + \frac{x^2}{2!}\delta^2 z_0$$
$$+ \frac{x(x^2-1)(x^2-4)}{5!\,2}(\delta^4 z_1 - \delta^4 z_{-1}) + \frac{x^2(x^2-1)}{4!}\delta^4 z_0$$
$$+ \frac{x(x^2-1)(x^2-4)(x^2-9)}{7!\,2}(\delta^6 z_1 - \delta^6 z_{-1}) + \frac{x^2(x^2-1)(x^2-4)}{6!}\delta^6 z_0$$
$$+ \text{etc.} \quad \text{.......................(xvii).}$$

Clearly each term in the central differences of z_1 and z_{-1} is of one higher difference order than the term in the same line in the central differences of z_0.

We can put this formula into a handier form for computing by using the relation

$$\delta^{2s} z_0 = \delta^{2s-2} z_1 + \delta^{2s-2} z_{-1} - 2\delta^{2s-2} z_0.$$

It then becomes:

$$z_x = (1-x)(1+x)z_0 + \tfrac{1}{2}x(1+x)z_1 - \tfrac{1}{2}x(1-x)z_{-1}$$
$$+\frac{x(x^2-1)}{4!}\{(x+2)\delta^2 z_1 + (x-2)\delta^2 z_{-1} - 2x\delta^2 z_0\}$$
$$+\frac{x(x^2-1)(x^2-4)}{6!}\{(x+3)\delta^4 z_1 + (x-3)\delta^4 z_{-1} - 2x\delta^4 z_0\}$$
$$+\frac{x(x^2-1)(x^2-4)(x^2-9)}{8!}\{(x+4)\delta^6 z_1 + (x-4)\delta^6 z_{-1} - 2x\delta^6 z_0\}$$
$$+\text{etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(xviii)}.$$

This is the mid-point central difference formula we have been seeking. We could of course get rid of all the z_0 central differences, but there are certain advantages of this result in its present form, although it requires us to use the central differences of three points instead of two as in Everitt's formula.

Consider the sixth order difference term; it may be written as

$$4(\delta^6 z_1 - \delta^6 z_{-1}) + x(\delta^6 z_1 + \delta^6 z_{-1} - 2\delta^6 z_0).$$

The first term is a seventh order difference, the second an eighth order difference. Hence if we neglect this term we shall be neglecting only seventh order differences. In other words our fourth order difference expression would neglect only seventh order differences. Everett's expression to fourth order differences neglects sixth order difference terms. Our expression is therefore more exact than his, but this exactness is gained at the expense of using $\delta^2 z_0$ and $\delta^4 z_0$ besides $\delta^2 z_1$, $\delta^2 z_{-1}$, $\delta^4 z_1$ and $\delta^4 z_{-1}$.

The mid-panel and mid-point central difference formulae seem to provide all that is requisite for interpolating into an existing table. Their high degree of accuracy enables a table to be constructed with fairly wide entry intervals provided δ^2 and δ^4 are tabulated. Indeed we should term a table satisfactory if in all stretches of it, we could interpolate to eight figure accuracy by aid of the tabled values of δ^2 and δ^4.

If the criticism be raised that table-users will not be content with formulae so cumbersome as (xi) and (xviii) (with side panel formulae like (viii) or (ix) for proximo-finial regions) we reply:

(*a*) That the formulae are largely cumbersome in appearance.

(*b*) That with the publication of tables of the central difference coefficients now in hand even (xviii) will reduce to

$$\begin{aligned} z_x = \; & f_1(x)z_0 + f_2(x)z_1 + f_3(x)z_{-1} \\ & + f_4(x)\delta^2 z_1 + f_5(x)\delta^2 z_{-1} + f_6(x)\delta^2 z_0 \\ & + f_7(x)\delta^4 z_1 + f_8(x)\delta^4 z_{-1} + f_9(x)\delta^4 z_0 \end{aligned}$$

where the factors $f(x)$ and the z quantities will both be tabled and the whole operation will be continuous on the machine.

(*c*) That experience in using such formulae for single entry tables is very desirable, for unless they are used in tables of double entry the bulk of such tables become impossible owing to the labour involved in computing and the expense of printing.

After all the computer of a table has to be considered as well as the user, and a balance struck between the two. The user often forgets the months and possibly years of work that have been spent to save him a few hours of labour. How limited is often the gratitude shown to men like Degen or Legendre, or the unknown men who before the machine age calculated tables of cube roots and of reciprocals! The user of a table groans if he has to use two differences, not stopping to ask how many the computer had to use in constructing the table. As a matter of fact the use of machines has largely replaced many of the old single difference tables. A table is not wanted twenty or more times an hour by the modern calculator. He computes his product 8 figures by 8 figures in far shorter time than it would take him to get merely an approximate result by logarithms. He rattles out square roots of ten figure numbers to any required number of figures on his machine with a rapidity quite beyond that of interpolation into a table of square roots. And yet though some tables are less frequently used, the coming age will be one of many tables. The paradox lies in the statement that the tables of the future will be tables made to save great *occasional* labour. They will be tables of integrals, of functions used in physical investigations, and of functions arising in statistics. Most of these are harder to calculate than the older logarithmic and trigonometrical functions. To start *de novo* and calculate such a function may need some hard thought and hard computing. The computer of such tables immensely eases the labour of the physicist and of the statistician; these latter workers must be content, if the computer has reduced their labour to as many minutes as they would have required hours without his aid. They must not grumble, if they are called upon in tables which they need only occasionally to put up with two differences. Ten tables of different functions with two central difference entries are of far greater value to the computer than a single function tabled to a tenth of the interval and so of the same bulk which can be entered with a single forward difference.

It is on the ground of even justice to table-user and table-computer that we appeal for a more liberal use of higher central difference interpolation formulae, and consequently for a wider knowledge of them, and a practical training in their use from the higher school classes onward.

We must destroy *ab initio* the current notion that to enter a table is to use proportional parts!

Lagrange's Formulae for filling in Frame Panels.

We have hitherto been dealing largely, though not entirely, with interpolation by central differences into completed tables. We have reached mid-panel and mid-point central difference formulae. These correspond to and are really identical—albeit in a different form—with Lagrangian parabolae passing through an even and an odd number of points respectively. The Lagrange formula is not a good one for interpolation into completed tables; it has marked superiority over difference formulae when we come to computing tables *de novo*.

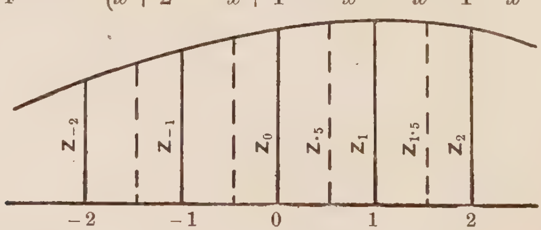
Given our framework we desire as a rule to halve the interval, to divide it into five equal sub-intervals, or to divide into ten equal sub-intervals. Other cases of course may occur, of which some will be referred to below, but these are the principal cases which are likely to arise, if the frame has been well chosen. On the other hand the fewer the frame entries used for interpolation, provided the requisite accuracy is obtainable, the lighter the labour. The maximum of accuracy will be attained when we interpolate in mid-panel or round mid-point, but since for proximo-finial regions it may be needful to use side panels these will be occasionally considered. We shall deal with the following cases:

(α) 5-point mid-point formula, and again: (ϵ) 4-point mid-panel formula,
(β) 7-point mid-point formula, (ζ) 6-point mid-panel formula,
(γ) 9-point mid-point formula, (η) 8-point mid-panel formula,
(δ) 11-point mid-point formula, (θ) 10-point mid-panel formula.

(α) *5-point Mid-point Formula.*

The general equation is:

$$z_x = \frac{x(x^2-1)(x^2-4)}{24}\left\{\frac{z_{-2}}{x+2} - 4\frac{z_{-1}}{x+1} + 6\frac{z_0}{x} - 4\frac{z_1}{x-1} + \frac{z_2}{x-2}\right\} \quad \ldots \ (\alpha)\,(\text{i}),$$



$$z_{\cdot 5} = \tfrac{1}{128}(3z_{-2} - 20z_{-1} + 90z_0 + 60z_1 - 5z_2) \quad \ldots\ldots\ldots\ldots (\alpha)\,(\text{ii}),$$
$$z_{1\cdot 5} = \tfrac{1}{128}(-5z_{-2} + 28z_{-1} - 70z_0 + 140z_1 + 35z_2) \quad \ldots\ldots\ldots\ldots (\alpha)\,(\text{iii}).$$

It is better to read the first value for example as above instead of

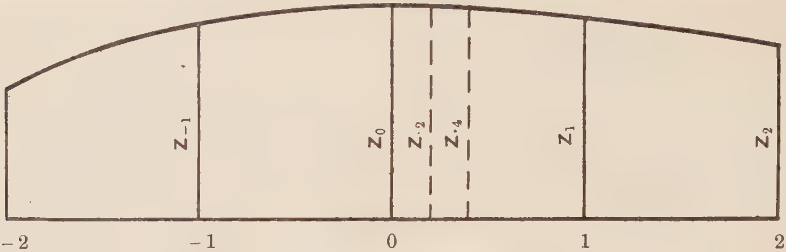
$$z_{\cdot5} = \cdot0234375 z_{-2} - \cdot15625 z_{-1} + \cdot703125 z_0 + \cdot46875 z_1 - \cdot0390625 z_2$$

because the operation of multiplying out by these longer numbers is slightly more laborious, than using the simpler multipliers and dividing by 128. Of course in both cases the operation is a continuous one on the machine. As the multipliers grow larger there is little difference in the labour, and we shall sometimes give one form and sometimes the other. An interchange of the negative and positive z's will of course give at once $z_{-\cdot5}$ and $z_{-1\cdot5}$.

A		
	(a) (ii)	
		·12330081
	+ 3	·12269642
	− 20	·12209793
	+ 90	·12150523
	⟶	[·1212,1103,4]
	+ 60	·12091826
	− 5	·12033692
		·11976114
	÷ 128	
B		

The method of working is illustrated at the side. AB is a slip of paper or thin card ruled to the spacing of the frame columns, with the multiplier opposite the corresponding frame entry. At top is the index number for storage and identification, at bottom the divisor. The arrow indicates the position of the interpolate. The continuous process on the machine gives ·1212,1103, which is correct to eight figures. Thus the number of entries in the table could be doubled by the process indicated. The slip AB is shifted gradually up or down the frame column, the dotted lines giving the interpolants and their multipliers, the arrow head the position of the corresponding interpolate. The process is precisely the same, if we were working with 11 interpolants, and inserting interpolates at $x = \cdot1, \cdot2, \cdot3, \cdot4$ and $\cdot5$ and at $x = -\cdot1, -\cdot2, -\cdot3, -\cdot4, -\cdot5$. In this case there will be nine vacant lines in the frame between the entries z_0 and z_1 and again between z_0 and z_{-1}. There will be *five* slips each with 11 multipliers on them and nine vacant lines between. The arrow head will point to the line for which the interpolate is calculated. The slips for $x = -\cdot1, -\cdot2, -\cdot3, -\cdot4, -\cdot5$ are obtained simply by reversing the order of the multipliers. It is a good test of the efficiency of the formula used to work $x = \cdot5$ first as a positive interpolate from z_0 and then as a negative one from z_1, and observe, if the two agree. In working a large table-frame by this slip process and continuous machining, the computer rapidly has the multipliers so impressed on his mind, that they pass mechanically on to the machine, the eye following only the frame entries.

We now divide up the frame interval into four equal parts. For this we only need to find $z_{\cdot 2}$ and $z_{\cdot 4}$, $z_{-\cdot 2}$ and $z_{-\cdot 4}$ will follow at once by reversal of slip.



We find : $z_{\cdot 2} = \frac{1}{625}(9z_{-2} - 66z_{-1} + 594z_0 + 99z_1 - 11z_2)$ ....(α) (iv),

$z_{\cdot 4} = \frac{1}{625}(14z_{-2} - 96z_{-1} + 504z_0 + 224z_1 - 21z_2)$ ......(α) (v).

	·12641419
(α) (v)	
+ 14	·12483812
− 96	·12330081
+ 504	·12180086
⟶	[·1212,1103,6]
+ 224	·1203,3692
− 21	·11890771
÷ 625	
	·11751201

Let us apply (α) (v) to interpolate on the same table as we used for the last illustration.

The interpolated value is ·1212,1103,6 only 2 in the ninth place of decimals above the value found from a frame which has entries four times as close. But as the actually calculated value is ·1212,1103, and we should be compelled to give our present result as ·1212,1104, we see that our frame is too open for a mid-point five-point interpolation to be quite satisfactory, unless we proposed to cut off at seven figures. Had we computed our frame to 9 or 10 figures this interpolation would very likely have been adequate*.

The Table from which these illustrations have been taken is that of the Trigamma Function computed by Miss Pairman†. In that case no interpolation was used and each entry was directly computed.

We can now proceed to the insertion of nine intermediate values by this formula. All we shall need will be $z_{\cdot 1}$ and $z_{\cdot 3}$, for the others have been already found.

* The illustration in the next paragraph suggests that this is the source of the slight discrepancy.

† *Tracts for Computers*, No. I. Cambridge University Press, 1919.

We find:

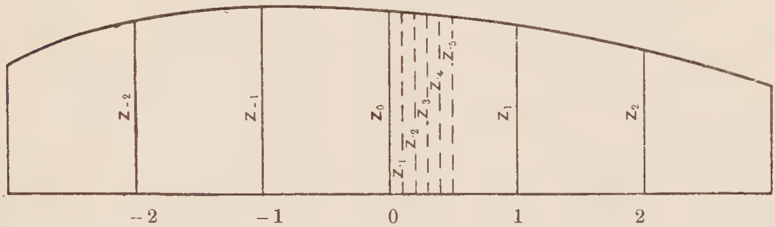
$$z_{\cdot 1}=\tfrac{1}{8}(\cdot 0627z_{-2}-\cdot 4788z_{-1}+7\cdot 9002z_0+\cdot 5852z_1-\cdot 0693z_2)\quad \ldots(\alpha)\,(\text{vi}),$$

$$z_{\cdot 2}=\tfrac{1}{625}(9z_{-2}-66z_{-1}+594z_0+99z_1-11z_2)\ \ldots\ldots\ldots\ldots\ldots(\alpha)\,(\text{iv}),$$

$$z_{\cdot 3}=\tfrac{1}{8}(\cdot 1547z_{-2}-1\cdot 0948z_{-1}+7\cdot 1162z_0+2\cdot 0332z_1-\cdot 2093z_2)\quad (\alpha)\,(\text{vii}),$$

$$z_{\cdot 4}=\tfrac{1}{625}(14z_{-2}-96z_{-1}+504z_0+224z_1-21z_2)\ \ldots\ldots\ldots\ldots\ldots(\alpha)\,(\text{v}),$$

$$z_{\cdot 5}=\tfrac{1}{128}(3z_{-2}-20z_{-1}+90z_0+60z_1-5z_2)\ \ldots\ldots\ldots\ldots\ldots(\alpha)\,(\text{ii}).$$



Let us apply this to find the function at 3·87 from the series:

x	
3·63	·12868867
3·73	·12546382
3·83	·12239644
→	
3·93	·11947530
4·03	·11669020

We find by (α) (v) $z_{3\cdot 87}=\cdot 1212{,}1102{,}9$ correct to 8 decimals.

It is clear therefore that this area of the table would be adequately based on frame values worked to nine or ten figures at intervals of ·1 and the ·01 values inserted by 5- or 6-point Lagrangian formulae.

(β) *7-point Mid-point Formula.*

The general equation is

$$z_x=\frac{x(x^2-1)(x^2-4)(x^2-9)}{6\,!}\left\{\frac{z_{-3}}{x+3}-6\frac{z_{-2}}{x+2}+15\frac{z_{-1}}{x+1}-20\frac{z_0}{x}\right.$$

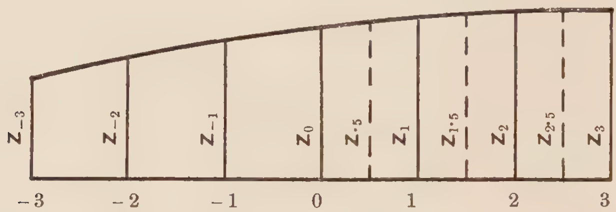
$$\left.+15\frac{z_1}{x-1}-6\frac{z_2}{x-2}+\frac{z_3}{x-3}\right\}\ldots\ldots(\beta)\,(\text{i}).$$

We first give mid-interval interpolation.

$$z_{\cdot 5} = \tfrac{1}{1024}\{-5z_{-3} + 42z_{-2} - 175z_{-1} + 700z_0 + 525z_1 - 70z_2 + 7z_3\} \;\ldots\ldots(\beta)\,(\text{ii}),$$

$$z_{1\cdot 5} = \tfrac{1}{1024}\{+7z_{-3} - 54z_{-2} + 189z_{-1} - 420z_0 + 945z_1 + 378z_2 - 21z_3\} \;\ldots(\beta)\,(\text{iii}),$$

$$z_{2\cdot 5} = \tfrac{1}{1024}\{-21z_{-3} + 154z_{-2} - 495z_{-1} + 924z_0 - 1155z_1 + 1386z_2 + 231z_3\} \;\ldots\ldots(\beta)\,(\text{iv}),$$



It will be seen that to obtain $z_{2\cdot 5}$ is like working with forward differences, and we may try the effect of such an interpolation on the following:

·1398,2863	z_{-3}
·1360,3040	z_{-2}
·1324,3277	z_{-1}
·1290,2028	z_0
·1257,7902	z_1
·1226,9642	z_2
→	← $z_{2\cdot 5}$
·1197,6114	z_3

We find $z_{2\cdot 5} =$ ·1212,1102,7 or in agreement to the 8th decimal place.

We may now try and see how this would work in the proximo-finial region of the Digamma Table *.

We have:

x			
·00	− ·5772,1566	z_{-3}	+ 231
→			
·08	− ·4527,9934	z_{-2}	+ 1386
·16	− ·3409,5315	z_{-1}	− 1155
·24	− ·2394,9368	z_0	+ 924
·32	− ·1467,4236	z_1	− 495
·40	− ·0613,8454	z_2	+ 154
·48	+ ·0176,2627	z_3	− 21

* *Tracts for Computers*, No. I.

We want $z_{-2\cdot5}$ and must reverse our multipliers. We deduce $z_{-2\cdot5} = -\cdot5132{,}7521$, but the true value is $z_{-2\cdot5} = \cdot5132{,}7488$.

Thus we should not be able to fill in this frame of ·08 intervals by the end panel values of a 7-point Lagrangian. If we reduce the frame to ·04 intervals or take:

·00	− ·5772,1566	z_{-3}	+ 231
·04	− ·5132,7488	z_{-2}	+ 1386
·08	− ·4527,9934	z_{-1}	− 1155
·12	− ·3954,5533	z_0	+ 924
·16	− ·3409,5315	z_1	− 495
·20	− ·2890,3990	z_2	+ 154
·24	− ·2394,9368	z_3	− 21

We find for $x = \cdot02$, $z_{-2\cdot5} = -\cdot5447{,}8931$ which is absolutely accurate.

Thus we see how in such a proximo-finial region of a table a formula which works well in the body of the table may fail, and we may need to much reduce our frame intervals.

We now give the formulae for dividing the mid-point space into four equal parts, i.e.

$$z_{\cdot2} = \tfrac{1}{78125}\{-231z_{-3} + 2016z_{-2} - 9240z_{-1} + 73920z_0 + 13860z_1 - 2464z_2 + 264z_3\} \ldots\ldots(\beta)\ (\text{v}),$$

$$z_{\cdot4} = \tfrac{1}{78125}\{-364z_{-3} + 3094z_{-2} - 13260z_{-1} + 61880z_0 + 30940z_1 - 4641z_2 + 476z_3\} \ldots\ldots(\beta)\ (\text{vi}).$$

We are approaching the limit at which much is practically gained by keeping the multipliers as the ratio of whole numbers.

To obtain the power of dividing the frame panel by a 7-point mid-point Lagrangian into tenth intervals we only need now to find $z_{\cdot1}$ and $z_{\cdot3}$. We have:

$$z_{\cdot1} = \frac{1}{80{,}000{,}000}\{-12{,}7281z_{-3} + 112{,}7346z_{\cdot2} - 538{,}0515z_{-1} + 7891{,}4220z_0 + 657{,}6185z_1 - 124{,}6014z_2 + 13{,}6059z_3\} \ldots\ldots(\beta)\ (\text{vii}),$$

or exactly:

$$z_{\cdot1} = -\cdot00159{,}10125z_{-3} + \cdot01409{,}18250z_{-2} - \cdot06725{,}64375z_{-1} + \cdot98642{,}77500z_0 + \cdot08220{,}23125z_1 - \cdot01557{,}51750z_2 + \cdot00170{,}07375z_3 \ldots\ldots(\beta)\ (\text{vii})^{\text{bis}}.$$

This form of the result presents little difficulty if the computer is using a 12 × 12 machine showing 24 figures in the product. Such a machine is a real necessity, if we work the frame to nine or ten decimal places. The older

8×9 machines showing 18 figures in the product may give some trouble; 8×9 machines showing 13 figures are quite inadequate.

$$z_{\cdot 3} = \frac{1}{80,000,000}\{-32,0229z_{-3} + 275,6754z_{-2} - 1219,3335z_{-1} + 7045,0380z_0 + 2264,4765z_1 - 372,9726z_2 + 39,1391z_3\} \ldots\ldots (\beta)\ (\text{viii}),$$

or

$$z_{\cdot 3} = -\cdot 00400,28625z_{-3} + \cdot 03445,94250z_{-2} - \cdot 15241,66875z_{-1} + \cdot 88062,97500z_0 + \cdot 28305,95625z_1 - \cdot 04662,15750z_2 + 00489,23875z_3 \ldots\ldots (\beta)\ (\text{viii})^{\text{bis}}.$$

It seems unnecessary to give special illustrations of the use of these formulae and to avoid taking up too much space we shall merely state the results for 9 and 11 mid-point Lagrangians. Sufficient illustrations of end and side panel interpolation formulae will be given in the cases of the even or mid-panel Lagrangians.

(γ) *9-point Mid-point Formula.*

The general equation is

$$z_x = \frac{x(x^2-1)(x^2-4)(x^2-9)(x^2-16)}{8\,!}\left\{\frac{z_{-4}}{x+4} - 8\frac{z_{-3}}{x+3} + 28\frac{z_{-2}}{x+2} - 56\frac{z_{-1}}{x+1} + 70\frac{z_0}{x} - 56\frac{z_1}{x-1} + 28\frac{z_2}{x-2} - 8\frac{z_3}{x-3} + \frac{z_4}{x-4}\right\} \ldots\ldots (\gamma)\ (\text{i}),$$

$$z_{\cdot 1} = \tfrac{1}{64}\{\cdot 021,983,247z_{-4} - \cdot 232,596,936z_{-3} + 1\cdot 201,750,836z_{-2} - 4\cdot 588,503,192z_{-1} + 63\cdot 091,918,890z_0 + 5\cdot 608,170,568z_1 - 1\cdot 328,250,924z_2 + \cdot 248,638,104z_3 - \cdot 023,110,593z_4\} \ldots\ldots (\gamma)\ (\text{ii}),$$

$$z_{\cdot 2} = \{\cdot 000,642,048z_{-4} - \cdot 006,741,504z_{-3} + \cdot 034,320,384z_{-2} - \cdot 125,841,408z_{-1} + \cdot 943,810,560z_0 + \cdot 188,762,112z_1 - \cdot 041,947,136z_2 + \cdot 007,704,576z_3 - \cdot 000,709,632z_4\} \ldots\ldots (\gamma)\ (\text{iii}),$$

$$z_{\cdot 3} = \tfrac{1}{64}\{\cdot 055,857,087z_{-4} - \cdot 582,267,816z_{-3} + 2\cdot 923,997,076z_{-2} - 10\cdot 346,451,192z_{-1} + 56\cdot 043,277,290z_0 + 19\cdot 214,837,928z_1 - 3\cdot 955,996,044z_2 + \cdot 711,660,664z_3 - \cdot 064,914,993z_4\} \ldots\ldots (\gamma)\ (\text{iv}),$$

$$z_{\cdot 4} = \{\cdot 001,018,368z_{-4} - \cdot 010,543,104z_{-3} + \cdot 052,276,224z_{-2} - \cdot 179,232,768z_{-1} + \cdot 784,143,360z_0 + \cdot 418,209,792z_1 - \cdot 078,414,366z_2 + \cdot 013,787,136z_3 - \cdot 001,244,672z_4\} \ldots\ldots (\gamma)\ (\text{v}),$$

$$z_{\cdot 5} = \tfrac{1}{512}\{\cdot 546,875z_{-4} - 5\cdot 625,000z_{-3} + 27\cdot 562,496z_{-2} - 91\cdot 875,000z_{-1} + 344\cdot 531,250z_0 + 275\cdot 625,000z_1 - 45\cdot 937,496z_2 + 7\cdot 875,000z_3 - \cdot 703,125z_4\} \ldots\ldots (\gamma)\ (\text{vi}),$$

$$z_{1\cdot5} = \tfrac{1}{512}\{-\cdot703{,}125z_{-4} + 6\cdot875{,}000z_{-3} - 30\cdot937{,}500z_{-2} + 86\cdot625{,}000z_{-1}$$
$$- 180\cdot468{,}750z_0 + 433\cdot125{,}000z_1 + 216\cdot562{,}500z_2 - 20\cdot625{,}000z_3$$
$$+ 1\cdot546{,}875z_4\} \ldots\ldots\ldots(\gamma)\,(\text{vii}),$$

$$z_{2\cdot5} = \tfrac{1}{512}\{1\cdot546{,}875z_{-4} - 14\cdot625{,}000z_{-3} + 62\cdot562{,}500z_{-2} - 160\cdot875{,}000z_1$$
$$+ 281\cdot531{,}250z_0 - 375\cdot375{,}000z_1 + 563\cdot062{,}500z_2 + 160\cdot875{,}000z_3$$
$$- 6\cdot703{,}125z_4\} \ldots\ldots\ldots(\gamma)\,(\text{viii}),$$

$$z_{3\cdot5} = \tfrac{1}{512}\{-6\cdot703{,}125z_{-4} + 61\cdot875{,}000z_{-3} - 255\cdot937{,}500z_{-2} + 625\cdot625{,}000z_1$$
$$- 1005\cdot468{,}750z_0 + 1126\cdot125{,}000z_1 - 938\cdot437{,}500z_2 + 804\cdot375{,}000z_3$$
$$+ 100\cdot546{,}875z_4\} \ldots\ldots\ldots(\gamma)\,(\text{ix}).$$

The multipliers are reduced to fewer figures by taking the divisor 512 in (γ) (vi) to (viii). Thus a 9×8 machine may be used*. No similar change suggests itself for $z_{\cdot1}$ and $z_{\cdot3}$. In all cases the multipliers are given exactly but the number of decimal places which it is desirable to retain in actual working will of course depend on the nature of the table which is being constructed.

(δ) 11-*point Mid-point Formula.*

The general equation is

$$z_x = \frac{x(x^2-1)(x^2-4)(x^2-9)(x^2-16)(x^2-25)}{10!}\left\{\frac{z_{-5}}{x+5} - 10\frac{z_{-4}}{x+4} + 45\frac{z_{-3}}{x+3}\right.$$
$$- 120\frac{z_{-2}}{x+2} + 210\frac{z_{-1}}{x+1} - 252\frac{z_0}{x} + 210\frac{z_1}{x-1} - 120\frac{z_2}{x-2} + 45\frac{z_3}{x-3}$$
$$\left. - 10\frac{z_4}{x-4} + \frac{z_5}{x-5}\right\} \ldots\ldots\ldots(\delta)\,(\text{i}),$$

$$z_{\cdot1} = \tfrac{1}{64}\{-\cdot0049{,}0714{,}9247z_{-5} + \cdot0610{,}4014{,}9170z_{-4} - \cdot3632{,}8733{,}9415z_{-3}$$
$$+ 1\cdot4300{,}8349{,}4840z_{-2} - 4\cdot7777{,}7894{,}8670z_{-1} + 63\cdot0666{,}8212{,}2441z_0$$
$$+ 5\cdot8395{,}0760{,}3930z_1 - 1\cdot5806{,}1859{,}9560z_2 + \cdot3883{,}4163{,}8685z_3$$
$$- \cdot0641{,}7041{,}3230z_4 + \cdot0051{,}0744{,}1053z_5\} \ldots\ldots\ldots(\delta)\,(\text{ii}),$$

$$z_{\cdot2} = \tfrac{1}{25}\{-\cdot00359{,}54688z_{-5} + \cdot04451{,}53280z_{-4} - \cdot26291{,}86560z_{-3}$$
$$+ 1\cdot01980{,}56960z_{-2} - 3\cdot27187{,}66080z_{-1} + 23\cdot55751{,}15776z_0$$
$$+ 4\cdot90781{,}49120z_1 - 1\cdot24642{,}91840z_2 + \cdot30047{,}84640z_3 - \cdot04920{,}11520z_4$$
$$+ \cdot00389{,}50912z_5\} \ldots\ldots\ldots(\delta)\,(\text{iii}),$$

* $z_{1\cdot5}$, $z_{2\cdot5}$ and $z_{3\cdot5}$ might be still further bettered by multiplying the factors and divisors by another 8.

$$z_{\cdot 3} = \tfrac{1}{64}\{-\cdot 0125{,}4301{,}9203 z_{-5} + \cdot 1546{,}0000{,}4130 z_{-4} - \cdot 9065{,}1820{,}6035 z_{-3}$$
$$+ 3\cdot 4684{,}1748{,}3960 z_{-2} - 10\cdot 7387{,}5413{,}3030 z_{-1} + 55\cdot 8415{,}2149{,}1756 z_0$$
$$+ 19\cdot 9434{,}0053{,}2770 z_1 - 4\cdot 6925{,}6483{,}1240 z_2 + 1\cdot 1079{,}6669{,}6265 z_3$$
$$- \cdot 1796{,}7027{,}5070 z_4 + \cdot 0141{,}4425{,}5697 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{iv}),$$

$$z_{\cdot 4} = \tfrac{1}{25}\{-\cdot 00572{,}54912 z_{-5} + \cdot 07026{,}73920 z_{-4} - \cdot 40920{,}42240 z_{-3}$$
$$+ 1\cdot 54588{,}26240 z_{-2} - 4\cdot 63764{,}78720 z_{-1} + 19\cdot 47812{,}10625 z_0$$
$$+ 10\cdot 82117{,}83680 z_1 - 2\cdot 31882{,}39360 z_2 + \cdot 53511{,}32160 z_3 - \cdot 08588{,}23680 z_4$$
$$+ \cdot 00672{,}12288 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{v}),$$

$$z_{\cdot 5} = \tfrac{1}{4096}\{-\cdot 984{,}375 z_{-5} + 12\cdot 031{,}250 z_{-4} - 69\cdot 609{,}375 z_{-3} + 259\cdot 875{,}000 z_{-2}$$
$$- 757\cdot 968{,}750 z_{-1} + 2728\cdot 687{,}500 z_0 + 2273\cdot 906{,}250 z_1 - 433\cdot 125{,}000 z_2$$
$$+ 97\cdot 453{,}125 z_3 - 15\cdot 468{,}750 z_4 + 1\cdot 203{,}125 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{vi}),$$

$$z_{1\cdot 5} = \tfrac{1}{4096}\{+ 1\cdot 203{,}125 z_{-5} - 14\cdot 218{,}750 z_{-4} + 78\cdot 203{,}125 z_{-3} - 268\cdot 125{,}000 z_{-2}$$
$$+ 656\cdot 906{,}250 z_{-1} - 1313\cdot 812{,}500 z_0 + 3284\cdot 531{,}250 z_1 + 1876\cdot 875{,}000 z_2$$
$$- 234\cdot 609{,}375 z_3 + 31\cdot 281{,}250 z_4 - 2\cdot 234{,}375 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{vii}),$$

$$z_{2\cdot 5} = \tfrac{1}{4096}\{- 2\cdot 234{,}375 z_{-5} + 25\cdot 781{,}250 z_{-4} - 137\cdot 109{,}375 z_{-3} + 446\cdot 875{,}000 z_{-2}$$
$$- 1005\cdot 468{,}750 z_{-1} + 1689\cdot 187{,}500 z_0 - 2346\cdot 093{,}750 z_1 + 4021\cdot 875{,}000 z_2$$
$$+ 1508\cdot 203{,}125 z_3 - 111\cdot 718{,}750 z_4 + 6\cdot 703{,}125 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{viii}),$$

$$z_{3\cdot 5} = \tfrac{1}{4096}\{+ 6\cdot 703{,}125 z_{-5} - 75\cdot 968{,}750 z_{-4} + 394\cdot 453{,}125 z_{-3} - 1243\cdot 125{,}000 z_{-2}$$
$$+ 2658\cdot 906{,}250 z_{-1} - 4102\cdot 312{,}500 z_0 + 4786\cdot 031{,}250 z_1 - 4558\cdot 125{,}000 z_2$$
$$+ 5127\cdot 890{,}625 z_3 + 1139\cdot 531{,}250 z_4 - 37\cdot 984{,}375 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{ix}),$$

$$z_{4\cdot 5} = \tfrac{1}{4096}\{-37\cdot 984{,}375 z_{-5} + 424\cdot 531{,}250 z_{-4} - 2165\cdot 109{,}375 z_{-3} + 6661\cdot 875{,}000 z_{-2}$$
$$- 13{,}777\cdot 968{,}750 z_{-1} + 20{,}207\cdot 687{,}500 z_0 - 21{,}651\cdot 093{,}750 z_1 + 17{,}320\cdot 875{,}000 z_2$$
$$- 10{,}825\cdot 546{,}875 z_3 + 7217\cdot 031{,}250 z_4 + 721\cdot 703{,}125 z_5\} \quad \ldots\ldots\ldots (\delta)\ (\text{x}).$$

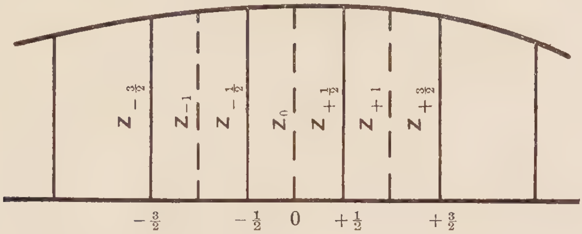
The results are arranged so as to give as few figures for machine multiplying as feasible, but we are reaching the limits of what can be done with existing machines in single operation. In many cases the reader will find it best to divide out by the denominator and retain only eight or ten figures of the quotients as may be requisite for the purpose he has in hand.

The above equations may prove useful in filling in a frame in proximo-finial areas, when the differential coefficients of the function take finite values. But the better policy is to reduce the frame-interval before interpolating, rather than to use 11-point interpolation formulae. If the tenth difference be not negligible then either the frame intervals are too large to make interpolation serviceable, or what is more important the differences are probably divergent.

This latter possibility is not directly manifest, when the Lagrangian method is applied, and this is its greatest defect.

We now take mid-panel Lagrangian systems and start with four ordinates.

(ε) 4-*point Mid-panel Lagrangian.*



The general formula is

$$z_x = \frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})}{3\,!}\left(-\frac{z_{-\frac{3}{2}}}{x+\frac{3}{2}}+3\frac{z_{-\frac{1}{2}}}{x+\frac{1}{2}}-3\frac{z_{+\frac{1}{2}}}{x-\frac{1}{2}}+\frac{z_{+\frac{3}{2}}}{x-\frac{3}{2}}\right) \ldots (\epsilon)\ (\text{i}).$$

For the mid-panel ordinates we have

$$z_0 = \tfrac{1}{16}\{9(z_{-\frac{1}{2}}+z_{+\frac{1}{2}})-(z_{-\frac{3}{2}}+z_{+\frac{3}{2}})\} \quad \ldots\ldots\ldots\ldots(\epsilon)\ (\text{ii}),$$

$$z_1 = \tfrac{1}{16}\{z_{-\frac{3}{2}}-5z_{-\frac{1}{2}}+15z_{+\frac{1}{2}}+5z_{+\frac{3}{2}}\} \quad \ldots\ldots\ldots(\epsilon)\ (\text{iii}),$$

$$z_{-1} = \tfrac{1}{16}\{5z_{-\frac{3}{2}}+15z_{-\frac{1}{2}}-5z_{+\frac{1}{2}}+z_{+\frac{3}{2}})\} \quad \ldots\ldots\ldots(\epsilon)\ (\text{iv}).$$

For four ordinates inserted in the centre panel, i.e. $z_{\cdot 1}$, $z_{\cdot 3}$, $z_{-\cdot 1}$ and $z_{-\cdot 3}$, we have:

$$z_{\cdot 1} = \{-{\cdot}056z_{-\frac{3}{2}}+{\cdot}448z_{-\frac{1}{2}}+{\cdot}672z_{+\frac{1}{2}}-{\cdot}064z_{+\frac{3}{2}}\} \quad \ldots\ldots(\epsilon)\ (\text{v}),$$

$$z_{\cdot 3} = \{-{\cdot}032z_{-\frac{3}{2}}+{\cdot}216z_{-\frac{1}{2}}+{\cdot}864z_{+\frac{1}{2}}-{\cdot}048z_{+\frac{3}{2}}\} \quad \ldots\ldots(\epsilon)\ (\text{vi}),$$

$$z_{-\cdot 1} = \{-{\cdot}064z_{-\frac{3}{2}}+{\cdot}672z_{-\frac{1}{2}}+{\cdot}448z_{+\frac{1}{2}}-{\cdot}056z_{+\frac{3}{2}}\} \ldots\ldots(\epsilon)\ (\text{vii}),$$

$$z_{-\cdot 3} = \{-{\cdot}048z_{+\frac{3}{2}}+{\cdot}864z_{-\frac{1}{2}}+{\cdot}216z_{+\frac{1}{2}}-{\cdot}032z_{+\frac{3}{2}}\} \quad \ldots(\epsilon)\ (\text{viii}).$$

For the insertion of four ordinates in the side panel we have to find $z_{\cdot 7}$, $z_{\cdot 9}$, $z_{1\cdot 1}$, $z_{1\cdot 3}$.

We obtain the values:

$$z_{\cdot 7} = \{{\cdot}032z_{-\frac{3}{2}}-{\cdot}176z_{-\frac{1}{2}}+1{\cdot}056z_{\frac{1}{2}}+{\cdot}088z_{\frac{3}{2}}\} \quad \ldots\ldots\ldots\ldots(\epsilon)\ (\text{ix}),$$

$$z_{\cdot 9} = \{{\cdot}056z_{-\frac{3}{2}}-{\cdot}288z_{-\frac{1}{2}}+1{\cdot}008z_{\frac{1}{2}}+{\cdot}224z_{\frac{3}{2}}\} \quad \ldots\ldots\ldots\ldots(\epsilon)\ (\text{x}),$$

$$z_{1\cdot 1} = \{{\cdot}064z_{-\frac{3}{2}}-{\cdot}312z_{-\frac{1}{2}}+{\cdot}832z_{\frac{1}{2}}+{\cdot}416z_{\frac{3}{2}}\} \quad \ldots\ldots\ldots\ldots(\epsilon)\ (\text{xi}),$$

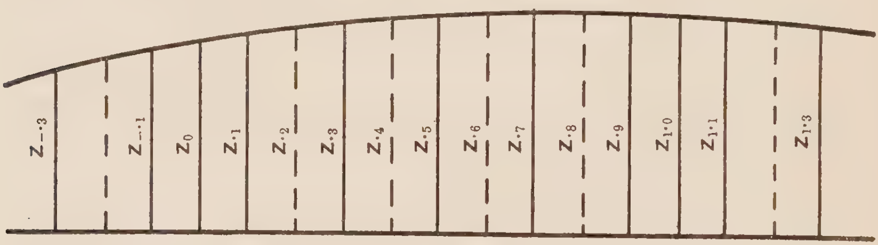
$$z_{1\cdot 3} = \{{\cdot}048z_{-\frac{3}{2}}-{\cdot}224z_{-\frac{1}{2}}+{\cdot}504z_{\frac{1}{2}}+{\cdot}672z_{\frac{3}{2}}\} \quad \ldots\ldots\ldots(\epsilon)\ (\text{xii}).$$

The values for $z_{-\cdot 7}$, $z_{-\cdot 9}$, $z_{-1\cdot 1}$ and $z_{-1\cdot 3}$ can be written down by reversion of the multipliers.

It will be noticed that by the processes just given we have found:

$$z_{-\cdot 1},\ z_0,\ z_{+\cdot 1} \text{ and again } z_{\cdot 3},\ z_{\cdot 5},\ z_{\cdot 7} \text{ and } z_{\cdot 9},\ z_{1\cdot 0},\ z_{1\cdot 1}.$$

Such system of nine ordinates the extremes being at single intervals and the three mid-ordinates at double intervals we have found advantageous as a stage towards breaking up frame panels into a tenth of their intervals. The entry values on either side of z_0 and z_1 may have been found either by interpolation or direct calculation as above. For example if the original frame is found to be too large for safe interpolation at a certain area of the table, we naturally proceed by direct computation to half the interval. This gives us our ·5 entries. Now if this half frame interval again proves too large in a further part of the table, say, a proximo-finial region, how shall we proceed? We know: 0, ·5, 1, 1·5, 2·0, etc.; we cannot very probably again halve as it would introduce unsuitable values of the argument, for example ·25, ·75, 1·25, etc., and we may desire to proceed by intervals of ·1. We can often halve the work of calculation by directly computing only the odd argument values, i.e. $-\cdot 1$, $+\cdot 1$, $+\cdot 3$, $+\cdot 7$, $+\cdot 9$, $+1\cdot 1$, etc., and then interpolating the even intervals. We have the following system:



There result:

$$z_{\cdot 2} = \tfrac{1}{3520}\{105z_{-\cdot 1} - 768z_0 + 2310z_{\cdot 1} + 2475z_{\cdot 3} - 924z_{\cdot 5} + 495z_{\cdot 7}$$
$$- 330z_{\cdot 9} + 192z_{1\cdot 0} - 35z_{1\cdot 1}\} \ldots\ldots\ldots\ldots\ldots(\omega)\,(\text{i}),$$

$$z_{\cdot 4} = \tfrac{1}{4224}\{-63z_{-\cdot 1} + 384z_0 - 770z_{\cdot 1} + 2475z_{\cdot 3} + 2772z_{\cdot 5} - 825z_{\cdot 7}$$
$$+ 462z_{\cdot 9} - 256z_{1\cdot 0} + 45z_{1\cdot 1}\} \ldots\ldots\ldots\ldots\ldots(\omega)\,(\text{ii}),$$

$$z_{\cdot 6} = \tfrac{1}{4224}\{45z_{-\cdot 1} - 256z_0 + 462z_{\cdot 1} - 825z_{\cdot 3} + 2772z_{\cdot 5} + 2475z_{\cdot 7}$$
$$- 770z_{\cdot 9} + 384z_{1\cdot 0} - 63z_{1\cdot 1}\} \ldots\ldots\ldots\ldots(\omega)\,(\text{iii}),$$

$$z_{\cdot 8} = \tfrac{1}{3520}\{-35z_{-\cdot 1} + 192z_0 - 330z_{\cdot 1} + 495z_{\cdot 3} - 924z_{\cdot 5} + 2475z_{\cdot 7}$$
$$+ 2310z_{\cdot 9} - 768z_{1\cdot 0} + 105z_{1\cdot 1}\} \ldots\ldots\ldots(\omega)\,(\text{iv}).$$

We recommend the computer after he has discovered that his frame in some special region of the table will not work to ·5 intervals, and he is

perplexed by the difficulty of wasting finally a calculated value if he computes at ·25, to compute the odd tenths and interpolate the even by the above formulae. We have found them work very satisfactorily in certain regions of the Incomplete Γ-function table, and for every four computed values: ·1, ·3, ·7 and ·9, four interpolated values were provided: ·2, ·4, ·6 and ·8. The close sets of terminal values thus seem to have a very good effect in controlling the Lagrangian high order parabola.

Discovery of Modes, Vertices and Maxima.

We propose to illustrate the use of four-point Lagrangian interpolation on the sort of problems which occur both in astronomy and ballistics. From four values obtained at equal intervals in the neighbourhood of the modal value of z it is desired to find the magnitude of that mode and its position.

Expanding (ϵ) (i) we deduce:

$$\begin{aligned} z_x = \frac{x^3}{3}&\left\{\tfrac{1}{2}(z_{\frac{3}{2}} - z_{-\frac{3}{2}}) - \tfrac{3}{2}(z_{\frac{1}{2}} - z_{-\frac{1}{2}})\right\} \\ &+ \frac{x^2}{2}\left\{\tfrac{1}{2}(z_{\frac{3}{2}} + z_{-\frac{3}{2}}) - \tfrac{1}{2}(z_{\frac{1}{2}} + z_{-\frac{1}{2}})\right\} \\ &- x\left\{\tfrac{1}{24}(z_{\frac{3}{2}} - z_{-\frac{3}{2}}) - \tfrac{9}{8}(z_{\frac{1}{2}} - z_{-\frac{1}{2}})\right\} \\ &- \left\{\tfrac{1}{16}(z_{\frac{3}{2}} + z_{-\frac{3}{2}}) - \tfrac{9}{16}(z_{\frac{1}{2}} + z_{-\frac{1}{2}})\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\psi)\ \text{(i)}. \end{aligned}$$

Differentiating we have to determine $\bar{x}$, the argument of the maximum or modal $\bar{z}$:

$$\begin{aligned} \bar{x}^2\left\{z_{\frac{3}{2}} - z_{-\frac{3}{2}} - 3(z_{\frac{1}{2}} - z_{-\frac{1}{2}})\right\} + \bar{x}\left\{z_{\frac{3}{2}} + z_{-\frac{3}{2}} - z_{\frac{1}{2}} - z_{-\frac{1}{2}}\right\} \\ - \tfrac{1}{12}\left\{z_{\frac{3}{2}} - z_{-\frac{3}{2}} - 27(z_{\frac{1}{2}} - z_{-\frac{1}{2}})\right\} \ldots\ldots(\psi)\ \text{(ii)}. \end{aligned}$$

Illustration A. To find position of vertex and time to vertex in a high-angled fire trajectory.

To clear times t we have the calculated positions:

t	Height in feet z		Horizonzal Distance in yards u		δ^2		δ^4
32	22917·408	$z_{-\frac{3}{2}}$	714·677				
				25·300			
33	22938·895	$z_{-\frac{1}{2}}$	739·977		·010		
→				25·310		·007	
34	22928·185	$z_{\frac{1}{2}}$	765·287		·017		·011
				25·327		·018	
35	22885·322	$z_{\frac{3}{2}}$	790·614		·035		
				25·36			
36			815·976				

Let $\bar{\tau}$ = time required from 33·5 seconds. Then by ψ (ii)

$$\bar{\tau}^2 \cdot 044 - \bar{\tau}\, 64{\cdot}350 - 21{\cdot}42367 = 0.$$

The suitable root of this is $\bar{\tau} = -{\cdot}3328$.

Hence time to vertex $= 33{\cdot}1672$ secs.

From (ψ) (i) $\quad \bar{z} = \frac{\bar{\tau}^3}{6} \cdot 044 - \frac{\bar{\tau}^2}{4}\, 64{\cdot}350 - \frac{\bar{\tau}}{2}\, 21{\cdot}42367 + 22937{\cdot}5619$

whence we find $\quad \bar{z} = 22939{\cdot}345.$

Lastly we can find $\bar{u}$ by the central difference formula

$$\bar{u}_{\bar{\tau}} = \phi u_0 + \theta u_1 - \frac{\theta\phi}{6}\{(\phi+1)(\delta^2 u_0) + (\theta+1)\,\delta^2 u_1\},$$

the fourth difference term being negligible. We have

$$\theta = {\cdot}5 + \bar{\tau} = {\cdot}1672 \text{ and } \phi = {\cdot}8328.$$

Whence we determine $\quad \bar{u} = 744{\cdot}207.$

Illustration B. It is required to find the speed in and slope of a trajectory from a knowledge of the position at four intervals of time preceding that at which speed is required.

Let u be any position coordinate and let us measure time from u_0. Then a four-point Lagrange gives us:

$$u_t = \tfrac{1}{6}u_0\,(t+1)(t+2)(t+3) - \tfrac{1}{2}u_{-1}\,t\,(t+2)(t+3) + \tfrac{1}{2}u_{-2}\,t\,(t+1)(t+3) - \tfrac{1}{6}u_{-3}\,t\,(t+1)(t+2)$$

$$= u_0\,(\ldots + \tfrac{11}{6}t + \ldots) - u_{-1}\,(\ldots + 3t + \ldots) + u_{-2}\,(\ldots + \tfrac{3}{2}t + \ldots) - u_{-3}\,(\ldots + \tfrac{1}{3}t + \ldots).$$

Hence

$$\left(\frac{du_t}{dt}\right)_{t=0} = \tfrac{1}{6}\,(11u_0 - 18u_1 + 9u_{-2} - 2u_3) \ldots\ldots\ldots\ldots (\psi)\ (\text{iii}).$$

For example

t	Height in feet z	Horizontal Distance in yards u
32	22917·408	565·323
31	22863·695	550·865
30	22777·606	536·722
29	22658·920	522·613

Thus:

$$\left(\frac{du}{dt}\right)_{t=0} = +\frac{88{\cdot}255}{6} = +14{\cdot}7092 \text{ ft. per sec.}$$

$$\left(\frac{dz}{dt}\right)_{t=0} = +\frac{225{\cdot}592}{6} = +37{\cdot}5987 \text{ ft. per sec.}$$

Hence: speed $= 40{\cdot}3735$ ft. per sec.

$$\text{Slope of trajectory at 32 seconds} = \tan\theta = \frac{225 \cdot 592}{88 \cdot 255}$$

$$= 2 \cdot 556{,}1384$$

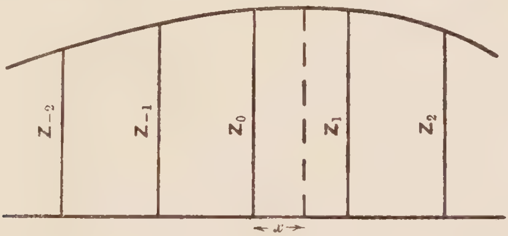
or,

$$\theta = 68^\circ\ 38' \cdot 029.$$

Illustration C. To find the maximum horizontal range and time of it from calculated ranges at equal intervals of quadrant elevation.

Quadrant Elevation (Q.E.)	Range in yards (R)		Time in seconds (t)	
30°	32934·704	z_{-2}	40·0645	t_1
35°	34226·410	z_{-1}	45·0172	t_2
40°	34842·717	z_0	49·6887	t_3
45°	34763·987	z_1	54·0672	t_4
50°	33963·467	z_2	58·1246	t_5

Very often in high-angle fire anti-aircraft range tables the 50° trajectory will not be carried to the ground as it demands excessive labour. The maximum range must then be determined by the four trajectories 30°, 35°, 40° and 45°. If the 50° be included, then we should obtain a cubic equation to solve in order to determine the Q.E. for which $dR/d\,(\text{Q.E.}) = 0$. There is no difficulty in solving a cubic with numerical coefficients but by actual testing we have found the following method, only involving a quadratic, as effective for practical purposes.



Let x be measured from z_0 to right then the Lagrangian through z_{-2}, z_{-1}, z_0 and z_1 is

$$z_x = \frac{x^3}{3}\left\{\tfrac{1}{2}(z_1 - z_{-2}) - \tfrac{3}{2}(z_0 - z_{-1})\right\} + \frac{x^2}{2}(z_1 + z_{-1} - 2z_0)\}$$

$$+ x\left\{\frac{z_{-1}}{3} + \frac{z_{-2}}{6} + \tfrac{1}{2}z_0 - z_{-1}\right\} + z_0 \quad \ldots\ldots\ldots\ldots\ldots (\psi)\ (\text{iv}),$$

giving on differentiation the quadratic to find $\bar{x}$ of the modal z:

$$\bar{x}^2\{z_1 - z_{-2} - 3(z_0 - z_{-1})\} + 2\bar{x}(z_1 + z_{-1} - 2z_0)$$

$$+ \tfrac{1}{3}\{2z_1 + z_{-2} + 3z_0 - 6z_{-1}\} = 0 \ldots\ldots\ldots (\psi)\ (\text{v}).$$

Now let us work from the other end and use z_2, z_1, z_0 and z_{-1}. It follows hat we must reverse the sign of x and exchange positive and negative subscripts. We find:

$$z_x = -\frac{x^3}{3}\{\tfrac{1}{2}(z_{-1}-z_2)-\tfrac{3}{2}(z_0-z_1)\}+\frac{x^2}{2}\{z_1+z_{-1}-2z_0\}$$
$$-x\left\{\frac{z_{+1}}{3}+\frac{z_2}{6}+\tfrac{1}{2}z_0-z_1\right\}+z_0 \;\ldots\ldots\ldots(\psi)\,(\text{vi}).$$

Taking the mean of (ψ) (iv) and (ψ) (vi) we obtain:

$$z_x = \frac{x^3}{12}\{z_2-z_{-2}-2(z_1-z_{-1})\}+\frac{x^2}{2}\{z_1+z_{-1}-2z_0\}$$
$$-\frac{x}{12}\{z_2-z_{-2}-8(z_1-z_{-1})\}+z_0 \ldots\ldots\ldots(\psi)\,(\text{vii}).$$

This leads to the quadratic for determining the modal value of x,

$$0=\bar{x}^2\{z_2-z_{-2}-2(z_1-z_{-1})\}+4x\{z_1+z_{-1}-2z_0\}$$
$$-\tfrac{1}{3}\{z_2-z_{-2}-8(z_1-z_{-1})\}\ldots\ldots\ldots\ldots(\psi)\,(\text{viii}).$$

A formula using *six* instead of five points may be obtained in much the same manner. If the points be z_1, z_2, z_3, z_4, z_5, z_6, we take the origin half way between z_3 and z_4 and write down by aid of (ψ) (vii) the value of z_x for z_1, z_2, z_3, z_4, z_5. Then we do the same thing for z_6, z_5, z_4, z_3, z_2, and take the mean of the two results.

The equation obtained is

$$\begin{aligned} z_x = \;& \frac{x^3}{24}\{-(z_1-z_6)+(z_2-z_5)+2(z_3-z_4)\} \\ &+\frac{x^2}{16}\{-(z_1+z_6)+7(z_2+z_5)-6(z_3+z_4)\} \\ &+\frac{x}{96}\{(z_1-z_6)-(z_2-z_5)-98(z_3-z_4)\} \\ &+\tfrac{1}{64}\{z_1+z_6-7(z_2+z_5)+38(z_3+z_4)\} \quad \ldots\ldots\ldots\ldots\ldots(\psi)\,(\text{ix}). \end{aligned}$$

Differentiating we obtain the quadratic:

$$\begin{aligned} &\bar{x}^2\{-(z_1-z_6)+(z_2-z_5)+2(z_3-z_4)\} \\ &\quad+\bar{x}\{-(z_1+z_6)+7(z_2+z_5)-6(z_3+z_4)\} \\ &\quad+\tfrac{1}{12}\{z_1-z_6-(z_2-z_5)-98(z_3-z_4)\}=0 \quad \ldots\ldots\ldots(\psi)\,(\text{x}). \end{aligned}$$

Or, more conveniently from the difference standpoint:

$$\begin{aligned} &\bar{x}^2\{z_2-z_1-2(z_4-z_3)+z_6-z_5\} \\ &\quad+\bar{x}\{z_2-z_1-6(z_3-z_2)-6(z_4-z_5)-(z_6-z_5)\} \\ &\quad+\tfrac{1}{12}\{-(z_2-z_1)+98(z_4-z_3)-(z_6-z_5)\}=0 \;\ldots(\psi)\,(\text{x})^{\text{bis}}. \end{aligned}$$

Writing this equation

$$a\bar{x}^2 + b\bar{x} + c = 0,$$

$\bar{z}$ may be computed from:

$$\bar{z} = \tfrac{1}{2}(z_3 + z_4) - \tfrac{1}{64}b + \tfrac{1}{12}c\bar{x} + \tfrac{1}{48}b\bar{x}^2 \quad \ldots\ldots\ldots\ldots (\psi)\,(\text{xi}).$$

Inverse Interpolation.

A somewhat similar artifice enables us to surmount some of the difficulties of inverse interpolation, i.e. finding x from a knowledge of z_x when differences higher than the second are essential. This would involve at least the solution of a cubic for the required value of the argument. But for most purposes we can reach quite adequate quadratics by the process of taking means as above.

For example, consider $z_{-\frac{3}{2}}, z_{-\frac{1}{2}}, z_{+\frac{1}{2}}, z_{+\frac{3}{2}}$. The parabola of the second order through $z_{-\frac{3}{2}}, z_{-\frac{1}{2}}, z_{+\frac{1}{2}}$ with the mid-point between $z_{-\frac{1}{2}}$ and $z_{\frac{1}{2}}$ for origin is

$$z_x = \tfrac{1}{4}x^2(z_{-\frac{3}{2}} + z_{\frac{1}{2}} - 2z_{-\frac{1}{2}}) + x(z_{\frac{1}{2}} - z_{-\frac{1}{2}}) + (-\tfrac{1}{8}z_{-\frac{3}{2}} + \tfrac{3}{4}z_{-\frac{1}{2}} + \tfrac{3}{8}z_{\frac{1}{2}}) \ldots (\psi)\,(\text{xii}).$$

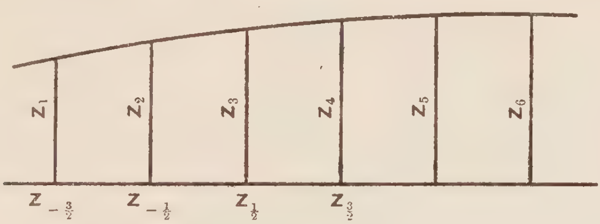
Put $-x$ for x and change the sign of the subscripts and we have the parabola through $z_{-\frac{1}{2}}$, $z_{\frac{1}{2}}$ and $z_{\frac{3}{2}}$. Add this to (ψ) (xii) and take the mean and we reach:

$$z_x = \frac{x^2}{4}\{(z_{\frac{3}{2}} + z_{-\frac{3}{2}}) - (z_{\frac{1}{2}} + z_{-\frac{1}{2}})\} + x(z_{\frac{1}{2}} - z_{-\frac{1}{2}})$$
$$+ \tfrac{1}{16}\{9(z_{\frac{1}{2}} + z_{-\frac{1}{2}}) - (z_{\frac{3}{2}} + z_{-\frac{3}{2}})\} \ldots\ldots\ldots (\psi)\,(\text{xiii}),$$

or

$$\tfrac{1}{4}x^2\{(z_{\frac{3}{2}} + z_{-\frac{3}{2}}) - (z_{\frac{1}{2}} + z_{-\frac{1}{2}})\} + x(z_{\frac{1}{2}} - z_{-\frac{1}{2}})$$
$$+ \tfrac{1}{16}\{9(z_{\frac{1}{2}} + z_{-\frac{1}{2}}) - (z_{\frac{3}{2}} + z_{-\frac{3}{2}})\} - z_x = 0 \ldots\ (\psi)\,(\text{xiv})$$

is a quadratic to find x when z_x is given. It is, perhaps, in a more useful form when we put the origin at $z_{-\frac{1}{2}}$ and then by taking a new nomenclature have the scheme:



Thus:

$$\tfrac{1}{4}(x - \tfrac{1}{2})^2(z_4 + z_1 - z_2 - z_3) + (x - \tfrac{1}{2})(z_3 - z_2) + \tfrac{1}{16}\{9(z_2 + z_3) - z_1 - z_4\} - z_x = 0,$$

or, $\tfrac{1}{4}x^2(z_1 - z_2 - z_3 + z_4) + \tfrac{1}{4}x(5z_3 - 3z_2 - z_1 - z_4) + z_2 - z_x = 0 \ldots\ldots (\psi)\,(\text{xv}),$

where x is measured from z_2*.

* If we put θ for x, u_{-1}, u_0, u_1, u_2, for z_1, z_2, z_3, z_4 respectively this is the inverse interpolation formula of the *Tables for Statisticians*, p. xiv, Equ. (vii)^{bis}.

If x be measured from z_3 the formula is

$$\tfrac{1}{4}x^2(z_1 - z_2 - z_3 + z_4) - \tfrac{1}{4}x(5z_2 - 3z_3 - z_1 - z_4) + z_3 - z_x = 0 \ldots\ldots (\psi)\,(\text{xv})^{\text{bis}}.$$

We can now change the sign of x and deduce the formula corresponding to this for z_5, z_4, z_3, z_2, i.e.

$$\tfrac{1}{4}x^2(z_5 - z_4 - z_3 + z_2) + \tfrac{1}{4}x(5z_4 - 3z_3 - z_5 - z_2) + z_3 - z_x = 0 \ldots\ldots (\psi)\,(\text{xv})^{\text{ter}}.$$

Taking the mean of the last two formulae, we obtain a quadratic interpolation formula for inverse work based on five entries, the origin of x being at the middle entry, namely:

$$\tfrac{1}{8}x^2(z_1 + z_5 - 2z_3) - \tfrac{1}{8}x\{z_5 - z_1 - 6(z_4 - z_2)\} + z_3 - z_x = 0 \ldots\ldots (\psi)\,(\text{xvi}).$$

It is not contended that this result is as exact as running a parabola of the fourth order through the five points, but inverse interpolation by that process would involve either repeated approximations or the solution of a biquadratic equation.

We may carry the process one stage further by writing $x = x + \frac{1}{2}$ so as to refer the origin to the midpoint of z_3 and z_4. We have:

$$\tfrac{1}{8}x^2\{z_1 + z_5 - 2z_3\} + \tfrac{1}{4}x\{z_1 - z_3 - 3(z_2 - z_4)\}$$
$$+ \tfrac{1}{32}\{3z_1 - 12z_2 + 30z_3 + 12z_4 - z_5\} - z_x = 0.$$

Changing sign of x and writing down the expression for z_6, z_5, z_4, z_3, z_2 and taking the mean we find:

$$\tfrac{1}{16}x^2\{z_1 + z_2 - 2(z_3 + z_4) + z_5 + z_6\} + \tfrac{1}{8}x\{z_1 - z_6 - 3(z_2 - z_5) - 4(z_3 - z_4)\}$$
$$+ \tfrac{1}{64}\{3(z_1 + z_6) - 13(z_2 + z_5) + 4z(z_3 + z_4)\} - z_x = 0 \ldots\ldots (\psi)\,(\text{xvii}).$$

This is the quadratic for a six-point inverse interpolation formula. The direct Lagrangian would need the solution of a quintic or repeated approximations.

We can now return to the problem of p. 35. Using four trajectories only we find from (ψ) (v):

$$19{\cdot}638\bar{x}^2 + 1390{\cdot}074\bar{x} - 544{\cdot}123 = 0,$$

of which the appropriate root is:

$$\bar{x} = {\cdot}387{,}025$$
$$= 1^\circ\, 56'{\cdot}105.$$

Thus the Q.E. of the maximum range is $41^\circ\, 56'{\cdot}105$. We will now determine the quadrant elevation of the maximum range, if we include the 50°

trajectory which was carried to the ground in this case. We must use (ψ) (viii). This gives:

$$46{\cdot}391\bar{x}^2 + 2780{\cdot}148\bar{x} - 1090{\cdot}617{,}668 = 0,$$

or, the appropriate root is

$$\bar{x} = {\cdot}389{,}753 = 1°\,56'{\cdot}926,$$

giving for Q.E. 41° 56′·926.

The quadrant elevation is thus lessened by 0′·82 only, if we include the range of the 50° trajectory in determining it.

Let us proceed to determine the maximum range on the two results from (ψ) (iv) and (ψ) (vii) respectively.

$$(\psi)\,(\text{iv})\text{ yields}: \quad R'_{\text{max.}} = 34895{\cdot}768 \text{ yds.}$$

$$(\psi)\,(\text{vii})\text{ yields}: \quad R_{\text{max.}} = 34895{\cdot}965 \text{ yds.}$$

Thus the difference of maximum range only amounts to 0·2 yds.

It remains to find the time to the ground for the maximum range. Working with central differences up to the fourth, i.e. (xi) of p. 14, with $\theta = {\cdot}389{,}753$ we find

$$\bar{t} = 51{\cdot}4311.$$

We might, however, have found $\bar{t}$ from (ψ) (xvi)

$$\bar{t} = t_3 + \tfrac{1}{8}\bar{x}^2 (t_1 + t_5 - 2t_3) - \tfrac{1}{8}\bar{x}\,\{t_5 - t_1 - 6\,(t_4 - t_2)\}$$

and this gives

$$\bar{t} = 51{\cdot}4317.$$

The difference is less than the thousandth of a second and has no importance in practical gunnery.

Interpolation at Unequal Intervals.

It will be noted that in this case we have found the time to the maximum range by way of the quadrant elevation, which proceeds by equal intervals. If we had simply been given the maximum range and the five other ranges with their times, we should have had the problem of interpolation at unequal intervals. The difficulties of the problem are surmounted in the present instance by the known fact that the ranges in this case correspond to equal intervals of quadrant elevation. Now we should have reached precisely the same results had we assumed, without knowing its nature at all, that there existed a third variable for which both our known variates would be values at equal intervals. This suggests a process of interpolation for values at unequal intervals: (i) use inverse interpolation into an assumed system at equal intervals, and (ii) then interpolate forwards into the second series.

We can illustrate this on a series where we are able to test at once the accuracy of our results. Find log 7·0631 from:

$$\log 7{\cdot}0005 = {\cdot}845{,}1291,$$
$$\log 7{\cdot}0277 = {\cdot}846{,}8132,$$
$$\log 7{\cdot}0502 = {\cdot}848{,}2014,$$
$$\log 7{\cdot}0691 = {\cdot}849{,}3641,$$
$$\log 7{\cdot}0846 = {\cdot}850{,}3153,$$
$$\log 7{\cdot}0965 = {\cdot}851{,}0442.$$

Work first with (ψ) (xvii) on the numbers of which we have given the log's above and we find for our quadratic

$$299x^2 - 3006x + 491{\cdot}75 = 0.$$

This gives

$$x = {\cdot}166{,}2846{,}374$$

as the fraction of the assumed variate at equal spacings, we now substitute this value of x in (ψ) (xvii) again, using the values of the logarithms for the z's, we then obtain

$$z_x = -\frac{{\cdot}001{,}8292}{16}x^2 + \frac{{\cdot}009{,}2420}{8}x + \frac{54{\cdot}3236004}{64}$$
$$= {\cdot}848{,}9954.$$

The true value to seven figures is ·848,9954. This is not bad, if we are only thinking of seven-figure values. If, however, we work with ten-figure logarithms instead of seven we find:

$$z_x = {\cdot}848{,}9952{,}2411,$$

as against the true value ·848,9953,551, or we see that the method will not give us a correct result in the seventh figure. We may need a more accurate process than this assumption of a pseudo-third variate at equal intervals*.

* Suppose it required to find the value of X that corresponds to Y, from a system x_1, x_2, x_3 ... corresponding to y_1, y_2, y_3 ... at unequal intervals. Let the quadratic to find the value of ϕ from mid-point of range of the hypothetical equal interval variate be

$$Y = a' + b'\phi + c'\phi^2 \text{ for } y, \quad \text{and} \quad X = a + b\phi + c\phi^2 \text{ for } x.$$

For example if we have four values only

$$a = \tfrac{1}{16}\{9(x_2 + x_3) - x_1 - x_4\}, \qquad a' = \tfrac{1}{16}\{9(y_2 + y_3) - y_1 - y_4\},$$
$$b = x_2 - x_3, \qquad b' = y_2 - y_3,$$
$$c = \tfrac{1}{4}\{x_1 + x_4 - x_2 - x_3\}, \qquad c' = \tfrac{1}{4}\{y_1 + y_4 - y_2 - y_3\}.$$

These will all be known; now find:

$$\lambda_x = b/c, \quad \lambda_y = b'/c', \quad \lambda = \lambda_x - \lambda_y,$$

and then determine

$$\mu_y = (Y - a')/\lambda c'.$$

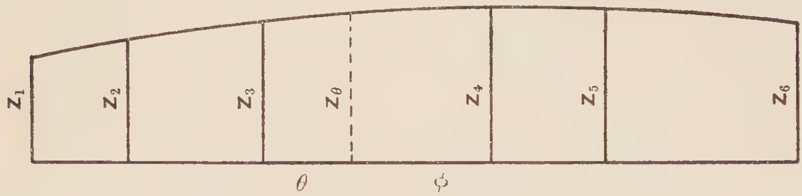
Solve the quadratic for μ_x

$$\mu_x{}^2 - (2\mu_y - \lambda_y)\,\mu_x + \mu_y(\mu_y - \lambda_x) = 0,$$

Interpolation at unequal argument intervals by differences.

Let us consider a system of six entries at unequal argument intervals, i.e. we can work with a central difference mid-point formula, which will be correct neglecting only six difference order terms.

We take:



If we take the mid-panel interval as our unit we have

$$z_\theta = \phi z_3 + \theta z_4 - \tfrac{1}{6}\theta\phi\{(\phi+1)\,\delta^2 z_3 + (\theta+1)\,\delta^2 z_4\}$$
$$+ \tfrac{1}{120}\theta(\theta+1)\,\phi(\phi+1)\{(\phi+2)\,\delta^4 z_3 + (\theta+2)\,\delta^4 z_4\}\ \ldots(\psi)\,(\text{xviii}).$$

Let $\zeta_s = z_s - \phi_s z_3 - \theta_s z_4$, then

$$\zeta_s = -\tfrac{1}{6}\theta_s\phi_s\{(\phi_s+1)\,\delta^2 z_3 + (\theta_s+1)\,\delta^2 z_4\}$$
$$+ \tfrac{1}{120}\theta_s(\theta_s+1)\,\phi_s(\phi_s+1)\{(\phi_s+2)\,\delta^4 z_3 + (\theta_s+2)\,\delta^4 z_4\}\ \ldots(\psi)\,(\text{xix}).$$

Now $\delta^2 z_3$, $\delta^2 z_4$, $\delta^4 z_3$ and $\delta^4 z_4$ are unknown, but if we put $s = 1, 2, 5$ and 6 in succession we have four equations to determine these four central differences. Their values can be found by determinants which reduce after some labour and provide the fairly simple system of results (ψ)(xx)—(ψ)(xxiii) on p. 42, which can then in any individual case be substituted in (xi).

choosing the appropriate root, and then

$$X = a + c\lambda\mu_x$$

gives the required value of X corresponding to Y. For example, taking the four mid-values of the system of logarithms on p. 42, we find:

$$a' = 7{\cdot}0600{,}8750, \qquad a = {\cdot}8488{,}1006{,}25,$$
$$b' = -{\cdot}0189{,}0000, \qquad b = {\cdot}0011{,}6270,$$
$$c' = -{\cdot}0017{,}5000, \qquad c = -{\cdot}0001{,}0928,$$
$$\lambda_y = 10{\cdot}8000{,}0000, \qquad \lambda_x = 10{\cdot}6425{,}6293,$$
$$\lambda = -{\cdot}1574{,}3707.$$

The quadratic for μ_x is

$$\mu_x^2 - 11{\cdot}0681{,}4794\,\mu_x + 3{\cdot}1874{,}0339 = 0,$$

giving: $$\mu_x = {\cdot}2958{,}9002 \quad\text{or}\quad 10{\cdot}7722{,}5792,$$

of which the latter is the appropriate root. We have

$$\lambda c\mu_x = {\cdot}0001{,}8528{,}28,$$

giving $$X = a + c\lambda\mu_x$$
$$= {\cdot}848{,}9953{,}453,$$

which differs from the true log 70631 a unit in the eighth figure. The result is as good as that obtained by the six-entry quadratic.

It may be asked: why not take a Lagrangian through the six-points and so interpolate?—The answer is two-fold. It would certainly be as easy to do so, if we had to interpolate one value only, but not if we had several to interpolate, and the central difference coefficients to table. Secondly by using the central difference formula we are able to judge whether our approximation has the requisite degree of accuracy; this cannot be determined when we use the Lagrangian method.

We have:

$$\delta^2 z_3 = \frac{2\zeta_1}{\theta_1\phi_1}\frac{(1+\theta_2)(1+\theta_5)(1+\theta_6)}{(\theta_1-\theta_2)(\theta_1-\theta_5)(\theta_1-\theta_6)} + \frac{2\zeta_2}{\theta_2\phi_2}\frac{(1+\theta_1)(1+\theta_5)(1+\theta_6)}{(\theta_2-\theta_1)(\theta_2-\theta_5)(\theta_2-\theta_6)}$$
$$+ \frac{2\zeta_5}{\theta_5\phi_5}\frac{(1+\theta_1)(1+\theta_2)(1+\theta_6)}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_6)} + \frac{2\zeta_6}{\theta_6\phi_6}\frac{(1+\theta_1)(1+\theta_2)(1+\theta_5)}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_5)} \ldots(\psi)\ (\text{xx}),$$

$$\delta^2 z_4 = \frac{2\zeta_1}{\theta_1\phi_1}\frac{(1+\phi_2)(1+\phi_5)(1+\phi_6)}{(\phi_1-\phi_2)(\phi_1-\phi_5)(\phi_1-\phi_6)} + \frac{2\zeta_2}{\theta_2\phi_2}\frac{(1+\phi_1)(1+\phi_5)(1+\phi_6)}{(\phi_2-\phi_1)(\phi_2-\phi_5)(\phi_2-\phi_6)}$$
$$+ \frac{2\zeta_5}{\theta_5\phi_5}\frac{(1+\phi_1)(1+\phi_2)(1+\phi_6)}{(\phi_5-\phi_1)(\phi_5-\phi_2)(\phi_5-\phi_6)} + \frac{2\zeta_6}{\theta_6\phi_6}\frac{(1+\phi_1)(1+\phi_2)(1+\phi_5)}{(\phi_6-\phi_1)(\phi_6-\phi_2)(\phi_6-\phi_5)} \ldots(\psi)\ (\text{xxi}),$$

$$\delta^4 z_3 = \frac{24\zeta_1}{\theta_1\phi_1}\frac{(\theta_2+\theta_5+\theta_6+1)}{(\theta_1-\theta_2)(\theta_1-\theta_5)(\theta_1-\theta_6)} + \frac{24\zeta_2}{\theta_2\phi_2}\frac{(\theta_1+\theta_5+\theta_6+1)}{(\theta_2-\theta_1)(\theta_2-\theta_5)(\theta_2-\theta_6)}$$
$$+ \frac{24\zeta_5}{\theta_5\phi_5}\frac{(\theta_1+\theta_2+\theta_6+1)}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_6)} + \frac{24\zeta_6}{\theta_6\phi_6}\frac{(\theta_1+\theta_2+\theta_5+1)}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_5)}$$
$$\ldots\ldots(\psi)\ (\text{xxii}),$$

$$\delta^4 z_4 = \frac{24\zeta_1}{\theta_1\phi_1}\frac{(\phi_2+\phi_5+\phi_6+1)}{(\phi_1-\phi_2)(\phi_1-\phi_5)(\phi_1-\phi_6)} + \frac{24\zeta_2}{\theta_2\phi_2}\frac{(\phi_1+\phi_5+\phi_6+1)}{(\phi_2-\phi_1)(\phi_2-\phi_5)(\phi_2-\phi_6)}$$
$$+ \frac{24\zeta_5}{\theta_5\phi_5}\frac{(\phi_1+\phi_2+\phi_6+1)}{(\phi_5-\phi_1)(\phi_5-\phi_2)(\phi_5-\phi_6)} + \frac{24\zeta_6}{\theta_6\phi_6}\frac{(\phi_1+\phi_2+\phi_5+1)}{(\phi_6-\phi_1)(\phi_6-\phi_2)(\phi_6-\phi_5)}$$
$$\ldots\ldots(\psi)\ (\text{xxiii}).$$

It will thus be seen that with a certain amount of computing labour we can determine the true central differences from unequally spaced ordinates.

Illustration. Provide an interpolation formula for logarithms between 7·0502 and 7·0691 from the data:

	Number	Logarithm	
x_1	7·0005	·845,1290,599	z_1
x_2	7·0277	·846,8132,140	z_2
x_3	7·0502	·848,2014,372	z_3
x_4	7·0691	·849,3641,253	z_4
x_5	7·0846	·850,3153,348	z_5
x_6	7·0965	·851,0442,071	z_6

and in particular find the logarithm of 7·0631.

We have: or, reduced to unit central panel,

$$x_2 - x_1 = 272, \qquad 1{\cdot}439{,}1534{,}392,$$
$$x_3 - x_2 = 225, \qquad 1{\cdot}190{,}4761{,}905,$$
$$x_4 - x_3 = 189, \qquad 1{\cdot}000{,}0000{,}000,$$
$$x_5 - x_4 = 155, \qquad {\cdot}820{,}1058{,}201,$$
$$x_6 - x_5 = 119, \qquad {\cdot}629{,}6296{,}296.$$

Thus we have:

$$\theta_1 = -2{\cdot}629{,}6296{,}296, \qquad \phi_1 = 3{\cdot}629{,}6296{,}296,$$
$$\theta_2 = -1{\cdot}190{,}4761{,}905, \qquad \phi_2 = 2{\cdot}190{,}4761{,}905,$$
$$\theta_5 = 1{\cdot}820{,}1058{,}201, \qquad \phi_5 = -{\cdot}820{,}1058{,}201,$$
$$\theta_6 = 2{\cdot}449{,}7354{,}497, \qquad \phi_6 = -1{\cdot}449{,}7354{,}497,$$

and:

$$\zeta_1 = -{\cdot}000{,}0149{,}381,$$
$$\zeta_2 = -{\cdot}000{,}0040{,}707,$$
$$\zeta_5 = -{\cdot}000{,}0023{,}178,$$
$$\zeta_6 = -{\cdot}000{,}0055{,}084.$$

Substituting in equations (ψ) (xx)—(xxiii), we find:

$$\delta^2 z_3 = -{\cdot}000{,}0031{,}210,$$
$$\delta^2 z_4 = -{\cdot}000{,}0031{,}046,$$
$$\delta^4 z_3 = -{\cdot}000{,}0000{,}003,$$
$$\delta^4 z_4 = +{\cdot}000{,}0000{,}005.$$

Fourth central differences are only sensible in the *tenth* figure, and therefore cannot be accurately determined from *ten*-figure logarithms.

To test these values let us take logarithms at 189 intervals: we find:

Number	Logarithm	δ^2	δ^4
6·9935			
7·0124	·845,8666,811		
7·0313	·847,0356,281	−31579	
7·0512	·848,2014,372	−31210	−4
7·0691	·849,3641,253	−31045	−1
7·0880	·850,5237,089	−30879	+2
7·1069	·851,6802,046	−30715	
7·1258	·852,8336,288		

Clearly our equations give excellent δ^2's and are not more irregular for δ^4's than might be expected from ten-figure logarithm tables. As no fourth

difference under 15 to 20 is significant to our ten figures we may omit them and the required formula runs:

$$z_\theta = \phi \times \cdot 848{,}2014{,}372 + \theta \times \cdot 849{,}3641{,}253$$
$$+ \tfrac{1}{6}\frac{\theta\phi}{10^{10}}\{(\phi+1)\,31210 + (\theta+1)\,31046\}.$$

For the special case of log 7·0631, we have

$$\theta = \tfrac{129}{189} = \cdot 682{,}5396{,}825,$$
$$\phi = \qquad \cdot 317{,}4603{,}175.$$

Hence:

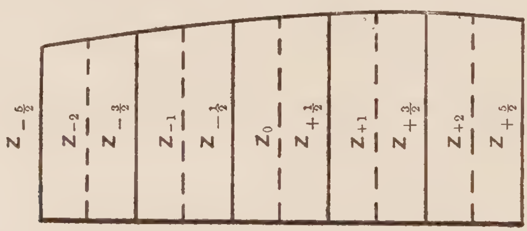
$$\log 7\cdot 0631 = \cdot 682{,}5396{,}825 \times \cdot 849{,}3641{,}253$$
$$+ \cdot 317{,}4603{,}175 \times \cdot 848{,}2014{,}372$$
$$+ \tfrac{1}{6}\cdot 682{,}5396{,}825 \times \cdot 317{,}4603{,}175 \times \frac{1}{10^{10}}$$
$$\times \{1\cdot 317{,}4603{,}175 \times 31210 + 1\cdot 682{,}5396{,}825 \times 31046\}$$
$$= \cdot 848{,}9953{,}551,$$

which is correct to the whole ten decimal places.

It will be seen that provided sixth differences are negligible, the method will give good results. The calculation of the pairs of second and fourth differences is, however, laborious, although it is clear that it is possible to arrange the computing so that a good deal of the work is the same for each*.

We now return to our even-entry Lagrangian interpolation formulae.

(ζ) *Six-Ordinate Mid-panel Lagrangian.*



The general formula is:

$$z_x = \frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})(x^2-\frac{25}{4})}{5\,!}\left\{-\frac{z_{-\frac{5}{2}}}{x+\frac{5}{2}} + 5\frac{z_{-\frac{3}{2}}}{x+\frac{3}{2}} - 10\frac{z_{-\frac{1}{2}}}{x+\frac{1}{2}}\right.$$
$$\left. + 10\frac{z_{\frac{1}{2}}}{x-\frac{1}{2}} - 5\frac{z_{\frac{3}{2}}}{x-\frac{3}{2}} + \frac{z_{\frac{5}{2}}}{x-\frac{5}{2}}\right\} \ldots(\zeta)\,(\text{i}).$$

For the mid-panel ordinates we have

* Note also that $\phi_2+\phi_5+\phi_6+1 = 5-(\theta_2+\theta_5+\theta_6+1)$ with similar results.

$$z_0 = \tfrac{1}{256}\{3(z_{-\frac{5}{2}} + z_{+\frac{5}{2}}) - 25(z_{-\frac{3}{2}} + z_{+\frac{3}{2}}) + 150(z_{-\frac{1}{2}} + z_{+\frac{1}{2}})\} \ldots (\zeta)\,(\text{ii}),$$

$$z_{-1} = \tfrac{1}{256}\{-7z_{-\frac{5}{2}} + 105z_{-\frac{3}{2}} + 210z_{-\frac{1}{2}} - 70z_{+\frac{1}{2}} + 21z_{+\frac{3}{2}} - 3z_{\frac{5}{2}}\} \ldots (\zeta)\,(\text{iii}),$$

$$z_{+1} = \tfrac{1}{256}\{-3z_{-\frac{5}{2}} + 21z_{-\frac{3}{2}} - 70z_{-\frac{1}{2}} + 210z_{+\frac{1}{2}} + 105z_{+\frac{3}{2}} - 7z_{\frac{5}{2}}\} \ldots (\zeta)\,(\text{iv}),$$

$$z_{-2} = \tfrac{1}{256}\{63z_{-\frac{5}{2}} + 315z_{-\frac{3}{2}} - 210z_{-\frac{1}{2}} + 126z_{+\frac{1}{2}} - 45z_{+\frac{3}{2}} + 7z_{+\frac{5}{2}}\} \ldots (\zeta)\,(\text{v}),$$

$$z_{+2} = \tfrac{1}{256}\{7z_{-\frac{5}{2}} - 45z_{-\frac{3}{2}} + 126z_{-\frac{1}{2}} - 210z_{+\frac{1}{2}} + 315z_{+\frac{3}{2}} + 63z_{+\frac{5}{2}}\} \ldots (\zeta)\,(\text{vi}).$$

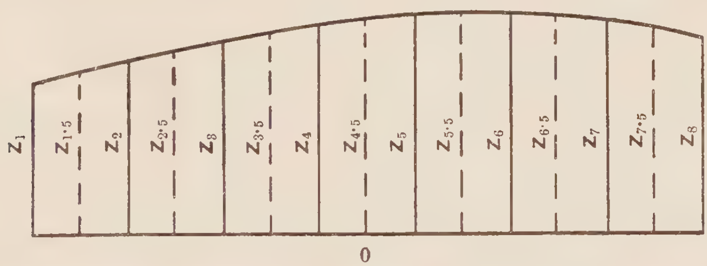
We give on accompanying sheet, Table A, formulae for the insertion of four entries between each of the six entries. It has been found more convenient in practice, however, to drop the fractional halves and use z_1, z_2, z_3, z_4, z_5, z_6 for the six entries, and present the coefficients in tabular form. If it be desirable to insert ten entries between each frame interval, then the equations (ζ) (ii)—(ζ) (vi) should first be used and afterwards the present system*. Of course if we are not working in a proximo-finial area of the frame only the mid-panel results (ζ) (ii) and (ζ) (xv)—(ζ) (xviii) should be applied. These formulae involving only six decimal places have been found very rapid and convenient in practice, as they can be applied in single operations with small machines and were found especially serviceable in both ballistic and bombing problems. They correspond to fifth difference accuracy, and therefore to central difference formulae in which second and fourth differences are used and six differences neglected.

(η) *Eight-Ordinate Mid-panel Lagrangian.*

The general formula is:

$$z_x = \frac{(x^2 - \frac{1}{4})(x^2 - \frac{9}{4})(x^2 - \frac{25}{4})(x^2 - \frac{49}{4})}{7\,!}\left\{-\frac{z_1}{x + \frac{7}{2}} + 7\frac{z_2}{x + \frac{5}{2}} - 21\frac{z_3}{x + \frac{3}{2}} + 35\frac{z_4}{x + \frac{1}{2}} - 35\frac{z_5}{x - \frac{1}{2}} + 21\frac{z_6}{x - \frac{3}{2}} - 7\frac{z_7}{x - \frac{5}{2}} + \frac{z_8}{x - \frac{7}{2}}\right\} \ldots (\eta)\,(\text{i}).$$

We consider first the insertion of mid-ordinates, i.e. we need $z_{1\cdot5}$, $z_{2\cdot5}$, $z_{3\cdot5}$, $z_{4\cdot5}$, the others follow by inversion of order of coefficients. These correspond to $x = -3$, $x = -2$, $x = -1$ and $x = 0$.



* An alternative method after the use of (ζ) (ii) and (ζ) (xv)—(ζ) (xviii) is to use the method of p. 32, i.e. (ω) (i)—(ω) (iv).

$$z_{4\cdot5} = \tfrac{1}{2048}\{1225(z_4+z_5) - 245(z_3+z_6) + 49(z_2+z_7) - 5(z_1+z_8)\} \ldots\ldots\ldots (\eta)\,(\text{ii}),$$

$$z_{3\cdot5} = \tfrac{1}{2048}\{9z_1 - 105z_2 + 945z_3 + 1575z_4 - 525z_5 + 189z_6 - 45z_7 + 5z_8\} \ldots (\eta)\,(\text{iii}),$$

$$z_{2\cdot5} = \tfrac{1}{2048}\{-33z_1 + 693z_2 + 2079z_3 - 1155z_4 + 693z_5 - 297z_6 + 77z_7 - 9z_8\} \ldots (\eta)\,(\text{iv}),$$

$$z_{1\cdot5} = \tfrac{1}{2048}\{429z_1 + 3003z_2 - 3003z_3 + 3003z_4 - 2145z_5 + 1001z_6 - 273z_7 + 33z_8\} \ldots\ldots (\eta)\,(\text{v}).$$

Equations (η) (vi) to (η) (xx) for inserting four equally spaced ordinates are given Table B of folding sheet.

(θ) *Ten-Ordinate Mid-panel Lagrangian.*

The general formula is:

$$z_x = \frac{(x^2-\frac{1}{4})(x^2-\frac{9}{4})(x^2-\frac{25}{4})(x^2-\frac{49}{4})(x^2-\frac{81}{4})}{9\,!}\left\{-\frac{z_1}{x+\frac{9}{2}} + 9\frac{z_2}{x+\frac{7}{2}} - 36\frac{z_3}{x+\frac{5}{2}} + 84\frac{z_4}{x+\frac{3}{2}} - 126\frac{z_5}{x+\frac{1}{2}} + 126\frac{z_6}{x-\frac{1}{2}} - 84\frac{x_7}{x-\frac{3}{2}} + 36\frac{z_8}{x-\frac{5}{2}} - 9\frac{z_9}{x-\frac{7}{2}} + \frac{z_{10}}{x-\frac{9}{2}}\right\} \ldots\ldots (\theta)\,(\text{i}).$$

Below are the formulae for inserting mid-ordinates in each panel:

$$z_{5\cdot5} = \tfrac{1}{65536}\{35(z_1+z_{10}) - 405(z_2+z_9) + 2268(z_3+z_8) - 8820(z_4+z_7) + 39690(z_5+z_6)\} \ldots\ldots (\theta)\,(\text{ii}),$$

$$z_{4\cdot5} = \tfrac{1}{65536}\{-55z_1 + 693z_2 - 4620z_3 + 32340z_4 + 48510z_5 + 16170z_6 + 6468z_7 - 1980z_8 + 385z_7 - 35z_{10}\} \ldots\ldots (\theta)\,(\text{iii}),$$

$$z_{3\cdot5} = \tfrac{1}{65536}\{143z_1 - 2145z_2 + 25740z_3 + 60060z_4 - 30030z_5 + 18018z_6 - 8580z_7 + 2860z_8 - 585z_9 + 55z_{10}\} \ldots\ldots (\theta)\,(\text{iv}),$$

$$z_{2\cdot5} = \tfrac{1}{65536}\{-715z_1 + 19305z_2 + 77220z_3 - 60060z_4 + 54054z_5 - 38610z_6 + 20020z_7 - 7020z_8 + 1485z_9 - 143z_{10}\} \ldots\ldots (\theta)\,(\text{v}),$$

$$z_{1\cdot5} = \tfrac{1}{5536}\{12155z_1 + 109395z_2 - 145860z_3 + 204204z_4 - 218790z_5 + 170170z_6 - 92820z_7 + 33660z_8 - 7293z_9 + 715z_{10}\} \ldots (\theta)\,(\text{vi}).$$

The general system of factors for inserting four equally spaced ordinates in each panel is given in the accompanying tabular sheet, Table C.

Of the formulae for 8- and 10-point Lagrangians it is not necessary, nor would it be profitable to give illustrations. They have been largely used in filling in the frame of our *Incomplete Γ-Function Tables.* The 8-point Lagrangian was used for the bulk, but as we approached low values of the argument (u) the 10-point had to be adopted or it would have been needful to reduce the frame-interval. As the whole operation is a continuous one on the machine, there is practically nothing to give by way of illustration except the answer, and the only question about the answer is: Is it correct to the required number of decimal places? This can only be tested by the criterion of direct computation, and it is this occasional test which tells us

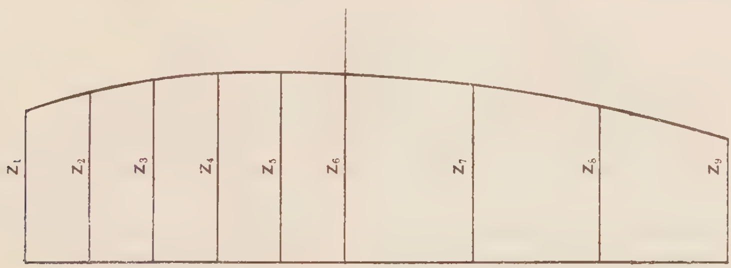
whether our Lagrangian is or is not of a high enough order. As we have already stated we consider a 10- or 11-point Lagrangian the limit to profitable continuous machine work. An experienced computer, we find, can use them effectively; but if nine or ten differences are inadequate, then it is better to reduce the frame-intervals, and use lower order Lagrangians.

On "Bridging" Formulae.

We define a "bridging" formula to be one which enables us to interpolate in the neighbourhood of an entry, where it has been thought advisable either on account of the economy of space or of labour to change the unit of the argument either in the case of a completed table or of a frame. In an extensive table, especially in a multi-variate table, economy of space demands that we should increase the argument of the entries if we reach an area of the table where an enlarged argument gives an adequately accurate interpolate with a fairly simple interpolation formula. In the case of the "frame" the computer of a table will naturally increase his argument to save labour, even if he proposes to complete his table throughout to the same argument interval, provided his larger interval gives the adequate accuracy of the interpolate.

It is at the boundary of an area, where this change of argument has been made, that certain difficulties of interpolation will result. We propose to illustrate methods of overcoming these difficulties in the present section.

If we suppose the larger argument interval to be a multiple of the smaller, there is no difficulty except that of excessive labour in taking the small interval to overlap the larger to such an extent that either large or small interval interpolation formula is safe for using in the overlapping zone. In a completed or published table this reduces to the provision in the neighbourhood of the boundary of *two* sets of, say, second and fourth central differences, for each entry in the immediate vicinity of the boundary. Such a duplication of differences enables the large interval formula to be used on one side and the small interval interpolation formula on the other side of the boundary, and hardly increases the labour of entering the table.



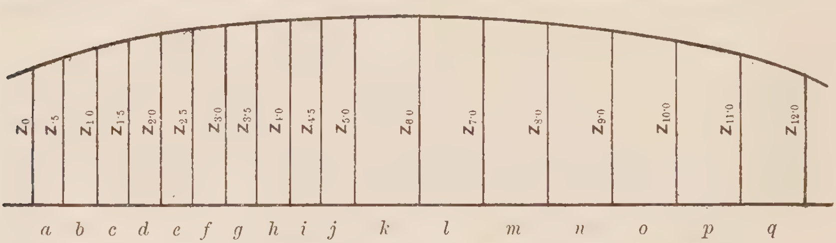
For example let z_6 denote a bounding ordinate, z_1, z_2, z_3, z_4, z_5, z_6 being entries at the smaller z_6, z_7, z_8, z_9 at the larger, say, double interval. Then if we are interpolating between z_5 and z_6 we need the second and fourth central differences of z_5 and z_6. But to ascertain these we must calculate $z_{6\cdot 5}$ and use z_7 to find the second and fourth differences of z_5 and z_6 on the smaller interval*. $z_{6\cdot 5}$ need not, however, be entered on the final completed table. To interpolate between z_6 and z_7 on the larger interval, we shall require the second and fourth differences of z_6 and z_7. These will require z_2 and z_4 to be used to ascertain these differences on the larger interval. Thus we shall be able to work either interpolation formula by providing double differences for z_6, these being calculated respectively from z_4, z_5, z_6, $z_{6\cdot 5}$, z_7 and z_2, z_4, z_6, z_7, z_8.

The matter is more complicated when we come to interpolate into a frame with changing interval, partly because it is not desirable to work by differences, and partly because we are using as a rule a higher difference formula. We will illustrate the procedure which we have adopted in the case of transition to a double argument interval in frame interpolation. We shall confine our attention to the use of an eight-entry Lagrangian, which we suppose to be an adequate formula on both sides of the boundary.

Suppose our arguments to be m plus the following system

0·0, 0·5, 1·0, 1·5, 2·0, 2·5, 3·0, 3·5, 4·0, 4·5, 5·0, 6·0, 7·0, 8·0, 9·0, 10·0, 11·0, 12·0

and let the corresponding values be given by z with the above subscripts omitting the part m of the argument. Then graphically we have:



We will suppose it needful to interpolate the values from $z_{0\cdot 0}$ to $z_{5\cdot 0}$ at 0·1 intervals and from $z_{5\cdot 0}$ to $z_{12\cdot 0}$ at 0·2 intervals.

Up to panel g from the left, unless we are in a proximo-finial region, we can work with the mid-panel eight-entry Lagrangian, namely (η) (xviii)—(xxi). Up to panel n from the right, we can work with the same mid-panel Lagrangian. It is the panels h, i, j, k, l, m which require special treatment.

* Or use the formulae given later for central differences to reduced intervals.

It would be bad policy to interpolate with *side*-panel 8-entry Lagrangians, because side-panel work is to be discarded unless we are driven to it by reaching a proximo-finial region. Accordingly we start with the axiom that we are to use mid-panel processes. Now the cautious computer is certain to have carried his smaller interval somewhat on the safe side and a mid-panel larger interval formula may be carried beyond n to the left. Thus we use, for

Panel m, $z_{4\cdot0}$, $z_{5\cdot0}$, $z_{6\cdot0}$, $z_{7\cdot0}$, $z_{8\cdot0}$, $z_{9\cdot0}$, $z_{10\cdot0}$, $z_{11\cdot0}$.

Panel l, $z_{3\cdot0}$, $z_{4\cdot0}$, $z_{5\cdot0}$, $z_{6\cdot0}$, $z_{7\cdot0}$, $z_{8\cdot0}$, $z_{9\cdot0}$, $z_{10\cdot0}$.

Panel k, $z_{2\cdot0}$, $z_{3\cdot0}$, $z_{4\cdot0}$, $z_{5\cdot0}$, $z_{6\cdot0}$, $z_{7\cdot0}$, $z_{8\cdot0}$, $z_{9\cdot0}$.

Should there be any doubt about this we may test $z_{4\cdot5}$ as the central ordinate of a next to mid-panel by (η) (iii) of p. 46, or by that equation reduced to present terminology, namely

$$z_{4\cdot5} = \tfrac{1}{2048}\{9z_2 - 105z_3 + 945z_4 + 1575z_5 - 525z_6 \\ + 189z_7 - 45z_8 + 5z_9\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(\eta)\,(\text{iii})^{bis}.$$

If this agrees with the directly calculated value, we may conclude that all is well.

An alternative to this process and also a good check on the process is to use the mid-panel formulae of an 8-entry Lagrangian using $z_{3\cdot5}$, $z_{4\cdot0}$, $z_{4\cdot5}$, $z_{5\cdot0}$, $z_{6\cdot0}$, $z_{7\cdot0}$, $z_{8\cdot0}$, and $z_{9\cdot0}$. These formulae are:

$$\begin{aligned}
z_{5\cdot2} = &- {\cdot}01584{,}77964^{*}z_{3\cdot5} + {\cdot}12155{,}13600z_{4\cdot0} - {\cdot}42336{,}25600z_{4\cdot5} \\
&+ 1{\cdot}21551{,}36000z_{5\cdot0} + {\cdot}12155{,}13600z_{6\cdot0} - {\cdot}02315{,}26400z_{7\cdot0} \\
&+ {\cdot}00413{,}44000z_{8\cdot0} - {\cdot}00038{,}772\dot{3}\dot{6}z_{9\cdot0}, \\
z_{5\cdot4} = &- {\cdot}02788{,}25891z_{3\cdot5} + {\cdot}20487{,}16800z_{4\cdot0} - {\cdot}64749{,}56800z_{4\cdot5} \\
&+ 1{\cdot}19508{,}48000z_{5\cdot0} + {\cdot}31868{,}92800z_{6\cdot0} - {\cdot}05121{,}79200z_{7\cdot0} \\
&+ {\cdot}00875{,}52000z_{8\cdot0} - {\cdot}00080{,}477\dot{0}\dot{9}z_{9\cdot0}, \\
z_{5\cdot6} = &- {\cdot}02970{,}9653\dot{3}z_{3\cdot5} + {\cdot}21111{,}55200z_{4\cdot0} - {\cdot}62390{,}27200z_{4\cdot5} \\
&+ {\cdot}93829{,}12000z_{5\cdot0} + {\cdot}56297{,}47200z_{6\cdot0} - {\cdot}06893{,}56800z_{7\cdot0} \\
&+ {\cdot}01117{,}0133\dot{3}z_{8\cdot0} - {\cdot}00100{,}35200z_{9\cdot0}, \\
z_{5\cdot8} = &- {\cdot}01947{,}35543^{\dagger}z_{3\cdot5} + {\cdot}13471{,}74400z_{4\cdot0} - {\cdot}37898{,}53257z_{4\cdot5} \\
&+ {\cdot}50519{,}04000z_{5\cdot0} + {\cdot}80830{,}46400z_{6\cdot0} - {\cdot}05773{,}60457z_{7\cdot0} \\
&+ {\cdot}00874{,}78857z_{8\cdot0} - {\cdot}00076{,}54400z_{9\cdot0},
\end{aligned}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\iota)\,(\text{i}).$$

* All figures are exact except this last 4 which is really $\dot{3}\dot{6}$.

† Coefficients ending in zero are exact; coefficients not ending in zero have long recurring decimals.

The formulae for the mid-panel 8-point Lagrangian using the larger unit throughout are for m:

$$\begin{aligned}
z_{5\cdot2} = &- {\cdot}0016{,}0512\,z_{2\cdot0} + {\cdot}0163{,}4304\,z_{3\cdot0} - {\cdot}0898{,}8672\,z_{4\cdot0} + {\cdot}8988{,}6720\,z_{5\cdot0}\\
&+ {\cdot}2247{,}1680\,z_{6\cdot0} - {\cdot}0599{,}2448\,z_{7\cdot0} + {\cdot}0128{,}4096\,z_{8\cdot0} - {\cdot}0013{,}5168\,z_{9\cdot0},\\
z_{5\cdot4} = &- {\cdot}0023{,}9616\,z_{2\cdot0} + {\cdot}0237{,}6192\,z_{3\cdot0} - {\cdot}1222{,}0416\,z_{4\cdot0} + {\cdot}7128{,}5760\,z_{5\cdot0}\\
&+ {\cdot}4752{,}3840\,z_{6\cdot0} - {\cdot}1069{,}2864\,z_{7\cdot0} + {\cdot}0219{,}3408\,z_{8\cdot0} - {\cdot}0022{,}6304\,z_{9\cdot0},\\
z_{5\cdot6} = &- {\cdot}0022{,}6304\,z_{2\cdot0} + {\cdot}0219{,}3408\,z_{3\cdot0} - {\cdot}1069{,}2864\,z_{4\cdot0} + {\cdot}4752{,}3840\,z_{5\cdot0}\\
&+ {\cdot}7128{,}5769\,z_{6\cdot0} - {\cdot}1222{,}0416\,z_{7\cdot0} + {\cdot}0237{,}6192\,z_{8\cdot0} - {\cdot}0023{,}9616\,z_{9\cdot0},\\
z_{5\cdot8} = &- {\cdot}0013{,}5168\,z_{2\cdot0} + {\cdot}0128{,}4096\,z_{3\cdot0} - {\cdot}0599{,}2448\,z_{4\cdot0} + {\cdot}2247{,}1680\,z_{5\cdot0}\\
&+ {\cdot}8988{,}6720\,z_{6\cdot0} - {\cdot}0898{,}8672\,z_{7\cdot0} + {\cdot}0163{,}4304\,z_{8\cdot0} - {\cdot}0016{,}0512\,z_{9\cdot0},
\end{aligned}$$

....(ι) (ii).

If it be desired to continue the 0·1 intervals beyond $z_{5\cdot0}$ we must use the additional formulae:

$$\begin{aligned}
z_{5\cdot1} = \tfrac{1}{8}\{ &- {\cdot}0070{,}9137\,z_{2\cdot0} + {\cdot}0732{,}7749\,z_{3\cdot0} - {\cdot}4196{,}8017\,z_{4\cdot0} + 7{\cdot}6941{,}3645\,z_{5\cdot0}\\
&+ {\cdot}8549{,}0405\,z_{6\cdot0} - {\cdot}2429{,}7273\,z_{7\cdot0} + {\cdot}0530{,}6301\,z_{8\cdot0} - {\cdot}0056{,}3673\,z_{9\cdot0}\},\\
z_{5\cdot3} = \tfrac{1}{8}\{ &- {\cdot}0169{,}2639\,z_{2\cdot0} + {\cdot}1699{,}9983\,z_{3\cdot0} - {\cdot}9023{,}0679\,z_{4\cdot0} + 6{\cdot}5166{,}6015\,z_{5\cdot0}\\
&+ 2{\cdot}7928{,}5435_{6\cdot0} - {\cdot}6899{,}9931\,z_{7\cdot0} + {\cdot}1448{,}1467\,z_{8\cdot0} - {\cdot}0150{,}9651\,z_{9\cdot0}\},\\
z_{5\cdot5} = \tfrac{1}{8}\{ &- {\cdot}0195{,}3125\,z_{2\cdot0} + {\cdot}1914{,}0625\,z_{3\cdot0} - {\cdot}9570{,}3125\,z_{4\cdot0} + 4{\cdot}7851{,}5625\,z_{5\cdot0}\\
&+ 4{\cdot}7851{,}5625\,z_{6\cdot0} - {\cdot}9570{,}3125\,z_{7\cdot0} + {\cdot}1914{,}0625\,z_{8\cdot0} - {\cdot}0195{,}3125\,z_{9\cdot0}\},\\
z_{5\cdot7} = \tfrac{1}{8}\{ &- {\cdot}0150{,}9651\,z_{2\cdot0} + {\cdot}1448{,}1467\,z_{3\cdot0} - {\cdot}6899{,}9931\,z_{4\cdot0} + 2{\cdot}7928{,}5435\,z_{5\cdot0}\\
&+ 6{\cdot}5166{,}6015\,z_{6\cdot0} - {\cdot}9023{,}0679\,z_{7\cdot0} + {\cdot}1699{,}9983\,z_{8\cdot0} - {\cdot}0169{,}2639\,z_{9\cdot0}\},\\
z_{5\cdot9} = \tfrac{1}{8}\{ &- {\cdot}0056{,}3673\,z_{2\cdot0} + {\cdot}0530{,}6301\,z_{3\cdot0} - {\cdot}2429{,}7273\,z_{4\cdot0} + {\cdot}8549{,}0405\,z_{5\cdot0}\\
&+ 7{\cdot}6941{,}3645\,z_{6\cdot0} - {\cdot}4196{,}8017\,z_{7\cdot0} + {\cdot}0732{,}7749\,z_{8\cdot0} - {\cdot}0070{,}9137\,z_{9\cdot0},
\end{aligned}$$

....(ι) (iii).

With the proper changes of the z's, all these formulae can of course be used in panels l and m.

We now turn to panel h, i and j, where true bridging formulae are requisite.

Panel h. *Formulae for Mid-panel Bridging 8-point Lagrangian**.

$$\begin{aligned}
z_{3\cdot6} = &- {\cdot}0002{,}2528\,z_{1\cdot5} + {\cdot}0116{,}1216\,z_{2\cdot5} - {\cdot}0827{,}9040\,z_{3\cdot0} + {\cdot}8941{,}3632\,z_{3\cdot5}\\
&+ {\cdot}2235{,}3408\,z_{4\cdot0} - {\cdot}0551{,}9360\,z_{4\cdot5} + {\cdot}0091{,}2384\,z_{5\cdot0} - {\cdot}0001{,}9712\,z_{6\cdot0},\\
z_{3\cdot7} = &- {\cdot}00034{,}0195\dot{5}\,z_{1\cdot5} + {\cdot}01683{,}96800\,z_{2\cdot5} - {\cdot}11226{,}4533\dot{3}\,z_{3\cdot0}\\
&+ {\cdot}70726{,}65600\,z_{3\cdot5} + {\cdot}47151{,}10400\,z_{4\cdot0} - {\cdot}09823{,}1466\dot{6}\,z_{4\cdot5}\\
&+ {\cdot}01554{,}43200\,z_{5\cdot0} - {\cdot}00032{,}5404\dot{4}\,z_{6\cdot0},
\end{aligned}$$

* All exact if account be taken of recurring decimals.

$$\begin{aligned} z_{3\cdot 8} = &- \cdot 000032{,}5404\dot{4}\, z_{1\cdot 5} + \cdot 01554{,}43200\, z_{2\cdot 5} - \cdot 09823{,}1466\dot{6}\, z_{3\cdot 0} \\ &+ \cdot 47151{,}10400\, z_{3\cdot 5} + \cdot 70726{,}65600\, z_{4\cdot 0} - \cdot 11226{,}4533\dot{3}\, z_{4\cdot 5} \\ &+ \cdot 01683{,}96800\, z_{5\cdot 0} - \cdot 00034{,}0195\dot{5}\, z_{6\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{3\cdot 9} = &- \cdot 0001{,}9712\, z_{1\cdot 5} + \cdot 0091{,}2384\, z_{2\cdot 5} - \cdot 0551{,}9360\, z_{3\cdot 0} + \cdot 2235{,}3408\, z_{3\cdot 5} \\ &+ \cdot 8941{,}3632\, z_{4\cdot 0} - \cdot 0827{,}9040\, z_{4\cdot 5} + \cdot 0116{,}1216\, z_{5\cdot 0} - \cdot 0002{,}2528\, z_{6\cdot 0}, \end{aligned}$$

....(ι) (iv).

Panel *i*. *Formulae for Mid-panel Bridging 8-point Lagrangian**.

$$\begin{aligned} z_{4\cdot 1} = &- \cdot 00014{,}65517\, z_{1\cdot 5} + \cdot 00261{,}96114\, z_{2\cdot 5} - \cdot 06287{,}06743\, z_{3\cdot 5} \\ &+ \cdot 88018{,}94400\, z_{4\cdot 0} + \cdot 22004{,}73600\, z_{4\cdot 5} - \cdot 04191{,}37828\, z_{5\cdot 0} \\ &+ \cdot 00220{,}59886\, z_{6\cdot 0} - \cdot 00013{,}13912\, z_{7\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 2} = &- \cdot 00022{,}15564\, z_{1\cdot 5} + \cdot 00387{,}07200\, z_{2\cdot 5} - \cdot 08460{,}28800\, z_{3\cdot 5} \\ &+ \cdot 69092{,}35200\, z_{4\cdot 0} + \cdot 46061{,}56800\, z_{4\cdot 5} - \cdot 07402{,}75200\, z_{5\cdot 0} \\ &+ \cdot 00365{,}56800\, z_{6\cdot 0} - \cdot 00021{,}36436\, z_{7\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 3} = &- \cdot 00021{,}36436\, z_{1\cdot 5} + \cdot 00365{,}56800\, z_{2\cdot 5} - \cdot 07402{,}75200\, z_{3\cdot 5} \\ &+ \cdot 46061{,}56800\, z_{4\cdot 0} + \cdot 69092{,}35200\, z_{4\cdot 5} - \cdot 08460{,}28800\, z_{5\cdot 0} \\ &+ \cdot 00387{,}07200\, z_{6\cdot 0} - \cdot 00022{,}15564\, z_{7\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 4} = &- \cdot 00013{,}13912\, z_{1\cdot 5} + \cdot 00220{,}59886\, z_{2\cdot 5} - \cdot 04191{,}37828\, z_{3\cdot 5} \\ &+ \cdot 22004{,}73600\, z_{4\cdot 0} + \cdot 88018{,}94400\, z_{4\cdot 5} - \cdot 06287{,}06743\, z_{5\cdot 0} \\ &+ \cdot 00261{,}96114\, z_{6\cdot 0} - \cdot 00014{,}65517\, z_{7\cdot 0}, \end{aligned}$$

....(ι) (v).

Panel *j*. *Formulae for Mid-panel Bridging 8-point Lagrangian†*.

$$\begin{aligned} z_{4\cdot 6} = &- \cdot 00031{,}24513\, z_{1\cdot 5} + \cdot 00359{,}7633\dot{3}\, z_{2\cdot 5} - \cdot 02518{,}3573\dot{3}\, z_{3\cdot 5} \\ &+ \cdot 83105{,}79200\, z_{4\cdot 5} + \cdot 20776{,}44800\, z_{5\cdot 0} - \cdot 01978{,}7093\dot{3}\, z_{6\cdot 0} \\ &+ \cdot 00314{,}7946\dot{6}\, z_{7\cdot 0} - \cdot 00028{,}48821\, z_{8\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 7} = &- \cdot 00046{,}26286\, z_{1\cdot 5} + \cdot 00524{,}87314\, z_{2\cdot 5} - \cdot 03528{,}31390\, z_{3\cdot 5} \\ &+ \cdot 63509{,}65029\, z_{4\cdot 5} + \cdot 42339{,}76685\, z_{5\cdot 0} - \cdot 03256{,}90514\, z_{6\cdot 0} \\ &+ \cdot 00502{,}05257\, z_{7\cdot 0} - \cdot 00044{,}86095\, z_{8\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 8} = &- \cdot 00044{,}86095\, z_{1\cdot 5} + \cdot 00502{,}05257\, z_{2\cdot 5} - \cdot 03256{,}90514\, z_{3\cdot 5} \\ &+ \cdot 42339{,}76685\, z_{4\cdot 5} + \cdot 63509{,}65029\, z_{5\cdot 3} - \cdot 03528{,}31390\, z_{6\cdot 0} \\ &+ \cdot 00524{,}87314\, z_{7\cdot 0} - \cdot 00046{,}26286\, z_{8\cdot 0}, \end{aligned}$$

$$\begin{aligned} z_{4\cdot 9} = &- \cdot 00028{,}48821\, z_{1\cdot 5} + \cdot 00314{,}7946\dot{6}\, z_{2\cdot 5} - \cdot 01978{,}7093\dot{3}\, z_{3\cdot 5} \\ &+ \cdot 20776{,}44800\, z_{4\cdot 5} + \cdot 83105{,}79200\, z_{5\cdot 0} - \cdot 02518{,}3573\dot{3}\, z_{6\cdot 0} \\ &+ \cdot 00359{,}7633\dot{3}\, z_{7\cdot 0} - \cdot 00031{,}24513\, z_{8\cdot 0}, \end{aligned}$$

....(ι) (vi).

* Only exact when last two figures are two zeros, otherwise to ten figures.

† These formulae, except when the last two digits are zero or the last digit recurs, are only exact to ten figures.

It remains to be seen how these formulae are obtained. *In every case we use a symmetrical system of intervals.* Thus for Panel h we use intervals $d+e, f, g, i, j, k$ or $z_{1\cdot5}, z_{2\cdot5}, z_{3\cdot0}, z_{3\cdot5}, z_{4\cdot0}, z_{4\cdot5}, z_{5\cdot0}, z_{6\cdot0}$. For Panel i we use $d+e, f+g, h, j, k, l$ or $z_{1\cdot5}, z_{2\cdot5}, z_{3\cdot5}, z_{4\cdot0}, z_{4\cdot5}, z_{5\cdot0}, z_{6\cdot0}, z_{7\cdot0}$. For Panel j we use $d+e, f+g, h+i, k, l, m$, or $z_{1\cdot5}, z_{2\cdot5}, z_{3\cdot5}, z_{4\cdot5}, z_{5\cdot0}, z_{6\cdot0}, z_{7\cdot0}, z_{8\cdot0}$. We do this because we believe better results are obtained from symmetry of intervals, than from crowded intervals on one side and large intervals on the other side of the panel under consideration. It is for this reason that we prefer our first method of treating Panel k to our second method.

The Calculation of Reduced Differences.

We have already indicated our preference for working with a high order Lagrangian in filling in a frame to working with a difference formula. It means that only one set of differences has to be computed, that of the final table for publication. This last set of differences also serves as a check on the computing. It is possible, however, to compute directly the differences on the reduced intervals from those on the larger intervals, and even if this method be not generally adopted (and we do not advise that it should be) it is useful for determining the value independently of a required—possibly suspected difference,—and thus serves for an occasional check.

As we have done throughout this tract, we lay most stress on central difference formulae, and shall here confine our attention to the calculation of reduced central differences.

The fundamental central difference formula for a single variate is (xi) or as we may write it:

$$z_\theta = \phi z_1 + \theta z_2 - \frac{\theta\phi}{3\,!}\{(\phi+1)\,\delta^2 z_1 + (\theta+1)\,\delta^2 z_2\}$$

$$+\frac{\theta(\theta+1)\,\phi(\phi+1)}{5\,!}\{(\phi+2)\,\delta^4 z_1 + (\theta+2)\,\delta^4 z_2\}$$

$$-\frac{\theta(\theta+1)(\theta+2)\,\phi(\phi+1)(\phi+2)}{7\,!}\{(\phi+3)\,\delta^6 z_1 + (\theta+3)\,\delta^6 z_2\} + \ldots (\kappa)\,(\text{i}),$$

where of course $\phi = 1-\theta$. Now write the coefficient of $\delta^{2s} z_1$ as $\Theta_{2s}(\theta)$ and that of $\delta^{2s} z_2$ as $\Phi_{2s}(\phi)$; both of these will be of the order $2s+1$ in θ or ϕ. Accordingly they may be expanded by Taylor's series with a finite number of terms. Thus if we wish to reduce to $\frac{1}{n}$th interval

$$\Theta_{2s}\left(\theta+\frac{1}{n}\right) = \Theta_{2s}(\theta) + \frac{1}{n}\,\Theta_{2s}'(\theta) + \frac{1}{2\,!}\,\frac{1}{n^2}\,\Theta_{2s}''(\theta) + \ldots \quad \ldots (\kappa)\,(\text{ii}),$$

and

$$\Theta_{2s}\left(\theta+\frac{1}{n}\right)+\Theta_{2s}\left(\theta-\frac{1}{n}\right)-2\Theta_{2s}(\theta)=\frac{2}{2!\,n^2}\Theta_{2s}''(\theta)+\frac{2}{4!\,n^4}\Theta_{2s}^{\text{iv}}(\theta)$$
$$+\frac{2}{6!\,n^6}\Theta_{2s}^{\text{vi}}(\theta)+\ldots\quad\ldots\ldots(\kappa)\,(\text{iii}),$$

$$\Phi_{2s}\left(\phi-\frac{1}{n}\right)+\Phi_{2s}\left(\phi+\frac{1}{n}\right)-2\Phi_{2s}(\phi)=\frac{2}{2!\,n^2}\Phi_{2s}''(\phi)+\frac{2}{4!\,n^4}\Phi_{2s}^{\text{iv}}(\phi)$$
$$+\frac{2}{6!\,n^6}\Phi_{2s}^{\text{vi}}(\phi)+\ldots\quad\ldots\ldots(\kappa)\,(\text{iv}).$$

Thus; $\partial^2 z_\theta = z_{\theta+\frac{1}{n}} + z_{\theta-\frac{1}{n}} - 2z_\theta$, where ∂^2 signifies the central difference of z_θ on $\frac{1}{n}$ the old interval, will be given by

$$\partial^2 z_\theta = -\frac{2}{2!\,n^2}\{\Theta_2''(\theta)\,\delta^2 z_1+\Phi_2''(\phi)\,\delta^2 z_2\}$$
$$+\left\{\left(\frac{2}{2!\,n^2}\Theta_4''(\theta)+\frac{2}{4!\,n^4}\Theta_4^{\text{iv}}(\theta)\right)\delta^4 z_1\right.$$
$$\left.+\left(\frac{2}{2!\,n^2}\Phi_4''(\phi)+\frac{2}{4!\,n^4}\Phi_4^{\text{iv}}(\phi)\right)\delta^4 z_2\right\}$$
$$-\left\{\left(\frac{2}{2!\,n^2}\Theta_6''(\theta)+\frac{2}{4!\,n^4}\Theta_6^{\text{iv}}(\theta)+\frac{2}{6!\,n^6}\Theta_6^{\text{vi}}(\theta)\right)\delta^6 z_1\right.$$
$$\left.+\left(\frac{2}{2!\,n^2}\Phi_6''(\phi)+\frac{2}{4!\,n^4}\Phi_6^{\text{iv}}(\phi)+\frac{2}{6!\,n^6}\Phi_6^{\text{vi}}(\phi)\right)\delta^6 z_2\right\}+\ldots\quad\ldots(\kappa)\,(\text{v}).$$

This is correct up to but not including eighth differences. It remains to find the differentials of Θ_{2s} and Φ_{2s}. These are straightforward algebra, and we have:

$$\Theta_2''(\theta)=-\phi,\qquad \Phi_2''(\phi)=-\theta,$$

$$\Theta_4''(\theta)=-\frac{\theta\phi(\phi+1)}{6}-\frac{\phi}{12},\qquad \Phi_4''(\phi)=-\frac{\theta\phi(\theta+1)}{6}-\frac{\theta}{12},$$

$$\Theta_4^{\text{iv}}(\theta)=\phi,\qquad \Phi_4^{\text{iv}}(\phi)=\theta,$$

$$\Theta_6''(\theta)=-\frac{\theta\phi(\theta+1)(\phi+1)(\phi+2)}{5!}-\frac{1}{12}\frac{\theta\phi(\phi+1)}{3!}-\tfrac{1}{90}\phi,$$

$$\Phi_6''(\phi)=-\frac{\theta\phi(\theta+1)(\phi+1)(\theta+2)}{5!}-\frac{1}{12}\frac{\theta\phi(\theta+1)}{3!}-\tfrac{1}{90}\theta,$$

$$\Theta_6^{\text{iv}}(\theta)=\frac{\theta\phi(\phi+1)}{3!}+\tfrac{1}{6}\phi,\qquad \Phi_6^{\text{iv}}(\phi)=\frac{\theta\phi(\theta+1)}{3!}+\tfrac{1}{6}\theta,$$

$$\Theta_6^{\text{vi}}(\theta)=-\phi,\qquad \Phi_6^{\text{vi}}(\phi)=-\theta.$$

Substituting we obtain:

$$\partial^2 z_\theta = \frac{\phi\delta^2 z_1 + \theta\delta^2 z_2}{n^2} - \frac{\phi}{6n^2}\left\{\theta(\phi+1) + \frac{1}{2}\left(1-\frac{1}{n^2}\right)\right\}\delta^4 z_1$$
$$- \frac{\theta}{6n^2}\left\{\phi(\theta+1) + \frac{1}{2}\left(1-\frac{1}{n^2}\right)\right\}\delta^4 z_2 + \frac{\phi}{n^2}\left\{\frac{\theta(\theta+1)(\phi+1)(\phi+2)}{120}\right.$$
$$\left. + \frac{\theta(\phi+1)}{72}\left(1-\frac{1}{n^2}\right) + \frac{1}{360}\left(1-\frac{1}{n^2}\right)\left(4-\frac{1}{n^2}\right)\right\}\delta^6 z_1$$
$$+ \frac{\theta}{n^2}\left\{\frac{\phi(\phi+1)(\theta+1)(\theta+2)}{120} + \frac{\phi(\theta+1)}{72}\left(1-\frac{1}{n^2}\right)\right.$$
$$\left. + \frac{1}{360}\left(1-\frac{1}{n^2}\right)\left(4-\frac{1}{n^2}\right)\right\}\delta^6 z_2$$

+ eighth order central difference terms ... .(κ) (vi).

Clearly this may be expressed as:

$$\partial^2 z_\theta = \frac{1}{n^2}\delta^2 z_\theta - \frac{1}{2}\frac{1}{3!}\frac{1}{n^2}\left(1-\frac{1}{n^2}\right)\delta^4 z_\theta + \frac{1}{3}\frac{1}{5!}\frac{1}{n^2}\left(1-\frac{1}{n^2}\right)\left(4-\frac{1}{n^2}\right)\delta^6 z_\theta - \dots$$

.........(κ) (vii),

a formula of the type given by Sheppard* but which involves a knowledge of the δ^2, δ^4, δ^6 etc. at the given point before we can find the ∂^2.

We can now proceed to the higher differences on the reduced interval. We have:

$$\partial^4 z_\theta = \delta^4 z_{\theta+\frac{1}{n}} + \delta^4 z_{\theta-\frac{1}{n}} - 2\delta^4 z_\theta,$$
$$\partial^6 z_\theta = \delta^6 z_{\theta+\frac{1}{n}} + \delta^6 z_{\theta-\frac{1}{n}} - 2\delta^6 z_\theta, \text{ etc.}$$

All linear terms in θ or ϕ will disappear and as our coefficients are again $\Theta_{2s}(\theta)$ and $\Phi_{2s}(\phi)$ we can readily write down the results. Thus:

$$\partial^4 z_\theta = \frac{\phi\delta^4 z_1 + \theta\delta^4 z_2}{n^4} - \frac{\phi}{6n^4}\left\{\theta(1+\phi) + \left(1-\frac{1}{n^2}\right)\right\}\delta^6 z_1$$
$$- \frac{\theta}{6n^4}\left\{\phi(\theta+1) + \left(1-\frac{1}{n^2}\right)\right\}\delta^6 z_2 + \dots \quad \text{......}(\kappa)\text{ (viii)},$$

$$\delta^6 z_\theta = \frac{\phi\delta^6 z_1 + \theta\delta^6 z_2}{n^6} + \dots \quad \text{..........}(\kappa)\text{ (ix)}.$$

The most useful cases are the subdivision of the frame interval into 5 and 10 intervals. In the first case all we need are $z_{1\cdot 2}$ and $z_{1\cdot 4}$ and their differences on the smaller interval. We have:

$$z_{1\cdot 2} = \cdot 8z_1 + \cdot 2z_2 - \cdot 048\,\delta^2 z_1 - \cdot 032\,\delta^2 z_2 + \cdot 008{,}064\,\delta^4 z_1$$
$$+ \cdot 006{,}336\,\delta^4 z_2 - \cdot 0016{,}0512\,\delta^6 z_1 - \cdot 0013{,}5168\,\delta^6 z_2 + \dots \quad \text{......}(\kappa)\text{ (x)},$$
$$z_{1\cdot 4} = 6z_1 + \cdot 4z_2 - \cdot 064\,\delta^2 z_1 - \cdot 056\,\delta^2 z_2 + \cdot 011{,}648\,\delta^4 z_1$$
$$+ \cdot 010{,}752\,\delta^4 z_2 - \cdot 0023{,}9616\,\delta^6 z_1 - \cdot 0022{,}6304\,\delta^6 z_2 + \dots \quad \text{......}(\kappa)\text{ (xi)}.$$

$z_{1\cdot 6}$ and $z_{1\cdot 8}$ will come by interchanging z_1 and z_2 in $z_{1\cdot 4}$ and $z_{1\cdot 2}$ respectively.

* *Proc. London Mathematical Society*, Vol. xxxi. p. 469, London, 1900.

The reader will see how doubtful is the gain of using differences over the 8-point Lagrangian. He has, it is true, less significant figures in both factors of his products, but he has to go through the laborious process of finding 6th (and possibly 8th or 10th) differences before be can get one factor of his products, and long before he has obtained these he will have turned the handle of his machine the requisite number of additional times in the continuous process.

We now write down the reduced differences:

$$\partial^2 z_{1\cdot 2} = \cdot 032\delta^2 z_1 + \cdot 008\,\delta^2 z_2 - \cdot 00448\,\delta^4 z_1 - \cdot 00192\,\delta^4 z_2$$
$$+ \cdot 0008{,}1408\,\delta^6 z_1 + \cdot 0004{,}4032\,\delta^6 z_2 - \dots \quad \dots\dots(\kappa)\,(\text{xii}),$$
$$\partial^4 z_{1\cdot 2} = \cdot 00128\,\delta^4 z_1 + \cdot 00032\,\delta^4 z_2 - \cdot 000{,}2816\,\delta^6 z_1 - \cdot 000{,}1024\,\delta^6 z_2 + \dots \quad \dots(\kappa)\,(\text{xiii}),$$
$$\partial^6 z_{1\cdot 2} = \cdot 000{,}0512\,\delta^6 z_1 + \cdot 000{,}0128\,\delta^6 z_2 - \dots \quad \dots\dots\dots\dots(\kappa)\,(\text{xiv}),$$

and again

$$\partial^2 z_{1\cdot 4} = \cdot 024\,\delta^2 z_1 + \cdot 016\,\delta^2 z_2 - \cdot 00448\,\delta^4 z_1 - \cdot 00352\,\delta^4 z_2$$
$$+ \cdot 0009{,}2416\,\delta^6 z_1 + \cdot 0007{,}7824\,\delta^6 z_2 - \dots \quad \dots\dots(\kappa)\,(\text{xv}),$$
$$\partial^4 z_{1\cdot 4} = \cdot 00096\,\delta^4 z_1 + \cdot 00064\,\delta^4 z_2 - \cdot 000{,}2560\,\delta^6 z_1 - \cdot 000{,}1920\,\delta^6 z_2 + \dots \quad \dots(\kappa)\,(\text{xvi}),$$
$$\partial^6 z_{1\cdot 4} = \cdot 000{,}0384\,\delta^6 z_1 + \cdot 000{,}0256\,\delta^6 z_2 - \dots \quad \dots\dots\dots\dots(\kappa)\,(\text{xvii}).$$

We will write down the formulae for a second case, namely that in which we subdivide the interval by 10, i.e. $n = 10$. We require to find $z_{1\cdot 1}$, $z_{1\cdot 2}$, $z_{1\cdot 3}$, $z_{1\cdot 4}$ and $z_{1\cdot 5}$. The remainder, i.e. $z_{1\cdot 6}$, $z_{1\cdot 7}$, $z_{1\cdot 8}$, $z_{1\cdot 9}$, will be obtained by interchanging z_1 and z_2 in $z_{1\cdot 4}$, $z_{1\cdot 3}$, $z_{1\cdot 2}$ and $z_{1\cdot 1}$ respectively.

Again $z_{1\cdot 2}$ and $z_{1\cdot 4}$ will have the same values as those given in (κ) (x)—(xi). Accordingly we need only $z_{1\cdot 1}$, $z_{1\cdot 3}$ and $z_{1\cdot 5}$ together with the central differences of all the five, $z_{1\cdot 1}$, $z_{1\cdot 2}$, $z_{1\cdot 3}$, $z_{1\cdot 4}$ and $z_{1\cdot 5}$ on the reduced intervals.

$$z_{1\cdot 1} = \cdot 9 z_1 + \cdot 1 z_2 - \cdot 0285\,\delta^2 z_1 - \cdot 0165\,\delta^2 z_2 + \cdot 0045{,}4575\,\delta^4 z_1$$
$$+ \cdot 0032{,}9175\,\delta^4 z_2 - \cdot 00088{,}642{,}125\,\delta^6 z_1 - \cdot 00070{,}459{,}125\,\delta^6 z_2 + \dots \ (\kappa)\,(\text{xviii}),$$
$$z_{1\cdot 3} = \cdot 7 z_1 + \cdot 3 z_2 - \cdot 0595\,\delta^2 z_1 - \cdot 0455\,\delta^2 z_2 + \cdot 0104{,}4225\,\delta^4 z_1$$
$$+ \cdot 0088{,}9525\,\delta^4 z_2 - \cdot 00211{,}579{,}875\,\delta^6 z_1 - \cdot 00188{,}706{,}375\,\delta^6 z_2 + \dots \quad (\kappa)\,(\text{xix}),$$
$$z_{1\cdot 5} = \cdot 5 z_1 + \cdot 5 z_2 - \cdot 0625\,\delta^2 z_1 - \cdot 0625\,\delta^2 z_2 + \cdot 0117{,}1875\,\delta^4 z_1$$
$$+ \cdot 0117{,}1875\,\delta^4 z_2 - \cdot 00244{,}140{,}625\,\delta^6 z_1 - \cdot 00244{,}140{,}625\,\delta^6 z_2 + \dots \ \dots(\kappa)\,(\text{xx}).$$

The following results give the central differences:

$$\partial^2 z_{1\cdot 1} = \cdot 009\,\delta^2 z_1 + \cdot 001\,\delta^2 z_2 - \cdot 001{,}0275\,\delta^4 z_1 - \cdot 000{,}2475\,\delta^4 z_2$$
$$+ \cdot 00016{,}77225\,\delta^6 z_1 + \cdot 00005{,}75025\,\delta^6 z_2 - \dots \quad (\kappa)\,(\text{xxi}),$$
$$\partial^4 z_{1\cdot 1} = \cdot 00009\,\delta^4 z_1 + \cdot 00001\,\delta^4 z_2 - \cdot 000{,}0177\,\delta^6 z_1 - \cdot 000{,}0033\,\delta^6 z_2 + \dots \quad (\kappa)\,(\text{xxii}),$$
$$\partial^6 z_{1\cdot 1} = \cdot 000{,}0009\,\delta^6 z_1 + \cdot 000{,}0001\,\delta^6 z_2 - \dots \quad \dots\dots\dots\dots(\kappa)\,(\text{xxiii}).$$

It will be observed that if our frame has been so chosen that sixth order central differences are adequate to interpolate to the $\frac{1}{10}$ intervals, then $\partial^6 z_{1\cdot 1}$ will hardly be significant.

Proceeding we have:

$$\partial^2 z_{1\cdot 2} = \cdot 008\,\delta^2 z_1 + \cdot 002\,\delta^2 z_2 - \cdot 001{,}1400\,\delta^4 z_1 - \cdot 000{,}4850\,\delta^4 z_2 + \cdot 00020{,}80200\,\delta^6 z_1 + \cdot 00011{,}17050\,\delta^6 z_2 - \dots \quad (\kappa)\,(\text{xxiv}),$$

$$\partial^4 z_{1\cdot 2} = \cdot 00008\,\delta^4 z_1 + \cdot 00002\,\delta^4 z_2 - \cdot 000{,}0180\,\delta^6 z_1 - \cdot 000{,}0065\,\delta^6 z_2 + \dots \quad (\kappa)\,(\text{xxv}),$$

$$\partial^6 z_{1\cdot 2} = \cdot 000{,}0008\,\delta^6 z_1 + \cdot 000{,}0002\,\delta^6 z_2 - \dots \dots\dots (\kappa)\,(\text{xxvi}).$$

$$\partial^2 z_{1\cdot 3} = \cdot 007\,\delta^2 z_1 + \cdot 003\,\delta^2 z_2 - \cdot 001{,}1725\,\delta^4 z_1 - \cdot 000{,}7025\,\delta^4 z_2 + \cdot 00023{,}03175\,\delta^6 z_1 + \cdot 00015{,}94075\,\delta^6 z_2 - \dots \quad (\kappa)\,(\text{xxvii}),$$

$$\partial^4 z_{1\cdot 3} = \cdot 00007\,\delta^4 z_1 + \cdot 00003\,\delta^4 z_2 - \cdot 000\cdot 0175\,\delta^6 z_1 - \cdot 000{,}0095\,\delta^6 z_2 + \dots \quad (\kappa)\,(\text{xxviii}),$$

$$\partial^6 z_{1\cdot 3} = \cdot 000{,}0007\,\delta^6 z_1 + \cdot 000{,}0003\,\delta^6 z_2 - \dots \dots\dots (\kappa)\,(\text{xxix}).$$

$$\partial^2 z_{1\cdot 4} = \cdot 006\,\delta^2 z_1 + \cdot 004\,\delta^2 z_2 - \cdot 001{,}1350\,\delta^4 z_1 - \cdot 000{,}8900\,\delta^4 z_2 + \cdot 00023{,}51150\,\delta^6 z_1 + \cdot 00019{,}76100\,\delta^6 z_2 - \dots \quad (\kappa)\,(\text{xxx}),$$

$$\partial^4 z_{1\cdot 4} = \cdot 00006\,\delta^4 z_1 + \cdot 00004\,\delta^4 z_2 - \cdot 000{,}0163\,\delta^6 z_1 - \cdot 000{,}0122\,\delta^6 z_2 + \dots \quad (\kappa)\,(\text{xxxi}),$$

$$\partial^6 z_{1\cdot 4} = \cdot 000{,}0006\,\delta^6 z_1 + \cdot 000{,}0004\,\delta^6 z_2 - \dots \dots\dots (\kappa)\,(\text{xxxii}).$$

$$\partial^2 z_{1\cdot 5} = \cdot 005\,\delta^2 z_1 + \cdot 005\,\delta^2 z_2 - \cdot 001{,}0375\,\delta^4 z_1 - \cdot 001{,}0375\,\delta^4 z_2 + \cdot 00022{,}36125\,\delta^6 z_1 + \cdot 00022{,}36125\,\delta^6 z_2 - \dots (\kappa)\,(\text{xxxiii}),$$

$$\partial^4 z_{1\cdot 5} = \cdot 00005\,\delta^4 z_1 + \cdot 00005\,\delta^4 z_2 - \cdot 000{,}0145\,\delta^2 z_1 - \cdot 000{,}0145\,\delta^2 z_2 + \dots \quad (\kappa)\,(\text{xxxiv}),$$

$$\partial^6 z_{1\cdot 5} = \cdot 000{,}0005\,\delta^6 z_1 + \cdot 000{,}0005\,\delta^6 z_2 - \dots \dots\dots (\kappa)\,(\text{xxxv}).$$

We provide these formulae as they may occasionally be useful, but we warn the reader that working by change of differences is more troublesome than using a high order Lagrangian.

APPENDIX I.

Illustration of the Efficacy of Central Difference Formulae by a Compressed Canon of Logarithms and Antilogarithms.

The table on p. 60 gives the logarithms of numbers from 10 to 100 and of antilogarithms from 0 to 100 proceeding by the unit, both to seven figures. The logarithms and antilogarithms are each followed by their first central differences. It is not generally recognised that complete tables of logarithms and antilogarithms could be readily reconstructed from these data by aid of the fundamental central difference formula:

$$z_\theta = \phi z_0 + \theta z_1 - \tfrac{1}{6}\theta\phi\{(\phi+1)\,\delta^2 z_0 + (\theta+1)\,\delta^2 z_1\}.$$

Let us deal first with the antilogarithmic canon. Now no fourth difference (∂^4) exceeds *3* in the whole of the canon. Hence the above formula will give the antilogarithm correct to seven figures*.

Illustration (i). Find the antilog of ·023,9644.

Here $\theta = \cdot 39644$, $\phi = \cdot 60356$ and we have to interpolate between ·02 and ·03. Hence reading the z's and their differences back from 7th figure†:

$$\begin{aligned} z_\theta &= 1047129 \times \cdot 60356 - \tfrac{1}{6}\cdot 39644 \times \cdot 60356\,\{1\cdot 60356 \times 555(3) \\ &\quad + 107159 \times \cdot 39644 \qquad\qquad\qquad + 1\cdot 39644 \times 568(1)\} \\ &= 1056798(0) - \cdot 03987922 \times 1683(77) \\ &= 1056798(0) - 67\,(1) = 1056731, \end{aligned}$$

which is correct to seven significant figures and only needs insertion of decimal point.

We will now try the other end of the table where the fourth differences are largest.

Illustration (ii). Find the antilogarithm of ·9853426.

Here: $\theta = \cdot 53426$, $\phi = \cdot 46574$, and accordingly

$$\begin{aligned} z_\theta &= \cdot 46574 \times 9549926 - \tfrac{1}{6}\cdot 53426 \times \cdot 46574\,\{1\cdot 46574 \times 5063 \\ &\quad + 53426 \times 9772372 \qquad\qquad\qquad + 1\cdot 53426 \times 5182\} \\ &= 9668770(0) - \cdot 04147104(2) \times 15371(6) \\ &= 9668770(0) - 637(47) \\ &= 9668132(5), \end{aligned}$$

* As a matter of fact I originally took my compressed canon to *eight* figures and the formula still gives the correct result to eight figures.

† It avoids much waste of space and even confusion to read the differences as they are taken from the Table and put on machine, without inserting rows of zeros or any decimal points.

which is correct to one unit in the eighth place {9668132(6)} and only needs the insertion of the decimal point.

It will be seen that our compressed canon of antilogarithms will provide by aid of the central difference formula antilogs correct to seven significant figures. In other words by taking out say to eight figures the antilogs of 00 to 100 we could by aid of δ^2 reconstruct the whole of the tables of either Shortrede or Filipowski (seven figure tables). To reconstruct Dodson's canon (11 figures) we should need to use δ^4 and initially our framework would need to consist of 12 figure antilogs. It is thus possible to use a frame of two figure logarithms to obtain a table of antilogarithms to seven figure accuracy.

We now turn to the table of logarithms.

Illustration (iii). Required the logarithm of 4·673218.

Here: $\theta = \cdot 73218$, $\phi = \cdot 26782$.

Accordingly

$$z_\theta = \cdot 26782 \times 6627578 + \cdot 73218 \times 6720979$$
$$- \frac{\cdot 73218 \times \cdot 26782}{6} \{1\cdot 26782\,(-2053) + 1\cdot 73218\,(-1966)\}$$
$$= 6695964(3) + \cdot 0326820746 \times 6008(3)$$
$$= 6695964(3) + 196(4) = 6696160(7),$$

which only differs from the correct value 6696160(4) by 3 in the eighth figure.

Now while the central difference formula to δ^2 (i.e. to third differences actually) will give seven figure correct results at the latter two-thirds of the table, it will fail in the first third, i.e. above the position (33) where the line is drawn across, unless we use a central difference formula including actually δ^4 terms. There are three ways of surmounting this difficulty, (i) we can actually calculate δ^4 from

$$\delta^4 z_0 = \delta^2 z_{+1} + \delta^2 z_{-1} - 2\delta^2 z_0,$$

(ii) We can use directly the formula (xii) of p. 15, i.e.

$$z_\theta = \phi z_0 + \theta z_1 - \tfrac{1}{120}\,\theta\phi\,(1+\phi)(2+\phi)(3\theta+8)\,\delta^2 z_0$$
$$- \tfrac{1}{120}\,\theta\phi\,(1+\theta)(2+\theta)(3\phi+8)\,\delta^2 z_1$$
$$+ \tfrac{1}{120}\,\theta\phi\,(1+\theta)(1+\phi)\,\{(2+\phi)\,\delta^2 z_{-1} + (2+\theta)\,\delta^2 z_2\} \ldots (\text{xii})^{bis},$$

thus appealing only to second differences already tabled.

(iii) We can by a multiplying factor reduce the logarithm to a part of the table where its value can be accurately found by δ^2 terms alone.

Illustration (iv). Find the logarithm of 1·056731.

We have taken purposely a number from the most difficult part of the table. We will illustrate all three methods.

(i) We have at once $\delta^4 z_0 = -2695$, $\delta^4 z_1 = -1131$, the $\delta^2 z_{-1}$ having been included in the compressed canon for this purpose. Thus by (xi), p. 14, since $\theta = \cdot 56731$, $\phi = \cdot 43269$, we have

$$z_\theta = \cdot 43269 \times 0 + \cdot 56731 \times 0413927 - \frac{\cdot 56731 \times \cdot 43269}{6} \{1\cdot 43269 \times (-43648) + 1\cdot 56731 \times (-\cdot 36041)\}$$

$$+ \frac{\cdot 56731 \times \cdot 43269}{6} \times \frac{1\cdot 43269 \times 1\cdot 56731}{20} \{2\cdot 43269 \times (-\cdot 2695) + 2\cdot 56731 \times (-1831)\}$$

$$= \cdot 0234824(9) + \cdot 04091156(1) \times 119021(5) - \cdot 04091156(1) \times \cdot 11227346(8) \times 11256(8)$$

$$= \cdot 0234824(9) + 4869(4) - 51(7) = \cdot 023{,}9642(6).$$

This leaves us a unit out in the seventh figure, the correct value being 0239644(5⁻).

(ii) We now turn to the second difference formula, substituting we have:

$$z_\theta = \cdot 56731 \times 0413927 - \frac{\cdot 56731 \times \cdot 43269}{120} \times 1\cdot 43269 \times 2\cdot 43269 \times 9\cdot 70193\,(-43648)$$

$$- \frac{\cdot 56731 \times \cdot 43269}{120} \times 1\cdot 56731 \times 2\cdot 56731 \times 9\cdot 29807\,(-36041)$$

$$+ \frac{\cdot 56731 \times \cdot 43269}{120} \times 1\cdot 56731 \times 1\cdot 43269 \{2\cdot 43269 \times (-53950) + 2\cdot 56731 \times (-30265)\}$$

$$= 0234824(9) + \cdot 002045578 \{1475915(5) + 1348412(8) - 469175(7)\}$$

$$= 023{,}9642(5) \text{ as before as it should do.}$$

Compressed Canon of Logarithms and Anti-logarithms.

	Logarithm	δ^2 −	Anti-logarithm	δ^2 +		Logarithm	δ^2 −	Anti-logarithm	δ^2 +
00	—	—	1000000	530	50	6989700	1738	3162278	1675
01	—	—	1023293	543	51	7075702	1670	3235937	1717
02	—	—	1047129	555	52	7160033	1606	3311311	1756
03	—	—	1071519	568	53	7242759	1546	3388442	1797
04	—	—	1096478	581	54	7323938	1490	3467369	1839
05	—	—	1122018	595	55	7403627	1436	3548134	1881
06	—	—	1148154	609	56	7481880	1385	3630781	1925
07	—	—	1174898	623	57	7558749	1337	3715352	1970
08	—	—	1202264	638	58	7634280	1291	3801894	2016
09	—	53950	1230269	652	59	7708520	1248	3890451	2063
10	0000000	43648	1258925	668	60	7781513	1207	3981072	2111
11	0413927	36041	1288250	683	61	7853298	1167	4073803	2160
12	0791812	30265	1318257	699	62	7923917	1130	4168694	2210
13	1139434	25774	1348963	715	63	7993405	1094	4265795	2262
14	1461280	22215	1380384	732	64	8061800	1060	4365158	2315
15	1760913	19345	1412538	749	65	8129134	1028	4466836	2368
16	2041200	16998	1445440	766	66	8195439	997	4570882	2424
17	2304489	15054	1479108	784	67	8260748	968	4677351	2480
18	2552725	13425	1513561	803	68	8325089	939	4786301	2538
19	2787536	12047	1548817	821	69	8388491	912	4897788	2597
20	3010300	10871	1584893	840	70	8450980	886	5011872	2657
21	3222193	9859	1621810	860	71	8512583	862	5128614	2719
22	3424227	8982	1659587	880	72	8573325	838	5248075	2783
23	3617278	8217	1698244	900	73	8633229	815	5370318	2847
24	3802112	7546	1737801	921	74	8692317	793	5495409	2914
25	3979400	6954	1778279	943	75	8750613	772	5623413	2982
26	4149733	6429	1819701	965	76	8808136	752	5754399	3051
27	4313638	5961	1862087	987	77	8864907	733	5888437	3122
28	4471580	5543	1905461	1010	78	8920946	714	6025596	3195
29	4623980	5167	1949845	1034	79	8976271	696	6165950	3269
30	4771213	4828	1995262	1058	80	9030900	679	6309573	3346
31	4913617	4522	2041738	1083	81	9084850	662	6456542	3423
32	5051500	4243	2089296	1108	82	9138139	646	6606934	3503
33	5185139	3990	2137962	1134	83	9190781	630	6760830	3585
34	5314789	3758	2187762	1160	84	9242793	616	6918310	3668
35	5440680	3547	2238721	1187	85	9294189	601	7079458	3754
36	5563025	3352	2290868	1215	86	9344985	587	7244360	3841
37	5682017	3174	2344229	1243	87	9395193	574	7413102	3931
38	5797836	3009	2398833	1272	88	9444827	561	7585776	4022
39	5910646	2856	2454709	1302	89	9493900	548	7762471	4116
40	6020600	2715	2511886	1332	90	9542425	536	7943282	4212
41	6127839	2584	2570396	1363	91	9590414	524	8128305	4310
42	6232493	2463	2630268	1395	92	9637878	513	8317638	4410
43	6334685	2349	2691535	1427	93	9684829	502	8511380	4513
44	6434527	2244	2754229	1460	94	9731279	492	8709636	4618
45	6532125	2145	2818383	1494	95	9777236	481	8912509	4726
46	6627578	2053	2884032	1529	96	9822712	471	9120108	4836
47	6720979	1966	2951209	1565	97	9867717	462	9332543	4948
48	6812412	1885	3019952	1601	98	9912261	452	9549926	5063
49	6901961	1809	3090295	1639	99	9956352	443	9772372	5182
50	6989700	1738	3162278	1675	100	10000000	434	10000000	5302

(iii) Multiply 1·056731 by 6 = 6·340386, and let us seek log 6·340386 which is in the latter two-thirds of the canon. Here

$$\theta = \cdot 40386, \quad \phi = \cdot 59614.$$

$$z_\theta = \cdot 59614 \times 7993405 + \cdot 40386 \times 8061800$$

$$- \frac{\cdot 40386 \times \cdot 59614}{6} \{1\cdot 59614 \times (-1094) + 1\cdot 40386 (-1060)\}$$

$$= 8021027(0) + \cdot 0401261834 \times 3234(3)$$

$$= 8021027(0) + 129(8) = 8021156(8).$$

But the value of $\log 6 = \cdot 778{,}1512(5)$.

Hence $\log 1\cdot 056731 = \cdot 023{,}96443$,

which is correct to the seventh figure and only two units wrong in the eighth.

We conclude therefore that this is the most accurate, as it is far the briefest, of the three processes.

The reader has accordingly before him the remarkable fact that a table of the logarithms of whole numbers from 34 to 100 if taken to eight figures is an adequate frame for finding all logarithms to seven figures of all seven figure numbers. The actual calculation of any such logarithm with a machine only needs about three minutes.

There is further knowledge to be gained from this table. Had we for example to form a table of logs and antilogs for all bases from 2 to 100, we should only need to multiply the present compressed canon about 100 fold, or it would be increased to the size of an ordinary seven figure table of logarithms. Such a table would suffice to provide in about five minutes by the bivariate central difference formula the log or antilog of any given seven figure argument to any of the given bases correct to seven significant figures.

This is really the key to multivariate tables, linear interpolation is dropped to reduce the entries to publishable proportions. But this does not mean necessarily lessened accuracy; it means that slightly longer time will be needful for interpolation.

APPENDIX II.

Some Works and Memoirs dealing with Interpolation.

(1) H. Briggs: *Arithmetica Logarithmica.* London, 1624. In Caput xiii, pp. 27—32 deal with the constancy of high differences (for example usually the fifth) in 15 figure logarithms and discusses the importance of the fact for the construction and testing of such tables. He points out (p. 29) that the same principle applies to trigonometrical tables. The same idea is developed in Caput xii, pp. 35—41 of H. Briggs' and H. Gellibrand's *Trigonometria Britannica*, Gouda, 1633, when Briggs is dealing with trigonometrical functions. There is no doubt that Briggs originated the use of high differences in the practical treatment, i.e. construction and interpolation of tables.

(2) J. Wallis: *Arithmetica Infinitorum.* Oxford, 1656. The first use of the words "interpolare" and "interpolatio" is attributed to Wallis: see for example, Props. clxx (p. 138), clxxv (p. 141), clxxxiv (p. 161), etc.

(3) G. Mouton: *Observationes diametrorum Solis Lunae apparentium*, etc. Lugduni, 1670. Liber iii, "De nonnullis numerorum proprietatibus," pp. 368—396, merely repeats Briggs' method of constructing tables by proceeding to a constant difference. There is no ground whatever for the continental practice of calling this process "Mouton's Method." He does not cite Briggs, but as he uses 14 figure logarithms for both mere numbers and trigonometrical functions, he must have been familiar with Briggs' works. He illustrates the method also on a few astronomical data. There is a copy of this work in the De Morgan Collection, University Library, London.

(4) I. Newton: *Philosophiae naturalis principia mathematica.* London, 1687, Book iii, Lemma 5 (following Prop. 40), pp. 481—483. Newton here gives for the first time what we now term the "forward difference" formula.

(5) I. Newton: *Methodus Differentialis*, pp. 93—101 of the *Analysis per Quantitatum, Series, Fluxiones ac Differentias....* London, 1711 (edited W. Jones, printed ex officina Pearsoniana). Gives the two formulae usually called Stirling's and Bessel's Formulae, first with unequal, then with equal argument intervals.

(5^{bis}) These formulae Stirling reproduces in his *Methodus differentialis, sive Tractatus de Summatione et Interpolatione Serierum Infinitarum.* London, 1730, pp. 104—106. In his *Phil. Trans.* paper, Vol. for 1718 (p. 1050) he credits Newton with them even in the title.

(6) D. Cotes: *Harmonia Mensurarum.* Cambridge, 1722. At the end of this work are printed among others two tracts by Cotes dealing with interpolation, namely *De methodo differentiali newtoniana* and *Canonotechnia, sive constructio Tabularum per Differentias**. The former somewhat extends Newton's method for running high order parabolae through points, and so finding areas of curves (integrals) by quadrature processes. The latter is an article on the construction of tables by inserting intermediate values by differences between the computed values. The propositions are numerous but the notation is at first unusual and troublesome.

(7) L. Euler: *Institutiones Calculi Differentialis.* Petrograd, 1755, Caput II, pp. 39—69, "De usu differentiarum in doctrina serierum" gives formulae which resemble Lagrangians. They are reached by considering series of which a certain order of difference is constant. The highest parabola Euler gives is one of the fifth order and his result coincides with our α (i), p. 22. He deduces it from Newton's forward difference formula. It is not, perhaps, an anticipation of Waring and Lagrange, but it is perilously near one.

(8) J. H. Lambert: *Beiträge zum Gebrauche der Mathematik*, Theil III, § v. *Anmerkungen über das Einschalten*, S. 66—104. Berlin, 1772. Nothing very suggestive or original about uni-variate interpolation but we have here—probably for the first time—bi-variate interpolation formulae, the chief being the forward difference formula in two dimensions (S. 94).

(9) E. Waring: "Problems concerning Interpolations." *Phil. Trans.* Vol. 69 (1779), pp. 59 *et seq.* London, 1779. Gives the formula usually attributed to Lagrange, and also in generalised forms. It is questionable whether he could possibly have taken his idea from an early issue of Euler's tract: see No. 10 however.

(10) L. Euler: *De eximio usu methodi interpolationum in serierum doctrina, Opuscula Analytica*, T. I, pp. 157—210. Petrograd, 1783. Gives (p. 184) the general Lagrangian for equal intervals. This does not antedate Waring unless the tract was originally printed at least four years earlier, but I have not found it in Reuss' *Repertorium*, nor in Poggendorff's *Handwörterbuch*, nor in the *R. S. Catalogue of Scientific Papers* as a separate publication. It does antedate Lagrange.

(11) J. L. Lagrange: "Leçons élementaires sur les Mathématiques. Leçon V. Sur l'Usage des Courbes dans la Solution des Problemes." *Journal de l'École polytechnique*, Tome II, Cahier 8, pp. 274—278. Paris, 1795. Gives the simple formula now called after Lagrange, but really due to Waring.

* They form part of the *Opera Miscellanea*, pp. 23—33 and 35—71. The reader must remember that Newton and his school signified by the "differential method" and "differences," our "finite differences," and used properly the separate term "fluxions" for our differential coefficients.

(12) G. S. KLÜGEL: *Mathematisches Wörterbuch*, Article "Einschalten," Band II, S. 10—56. Leipzig, 1805. Also Grunert's *Supplement* to the *Wörterbuch*, "Einschalten," Bd. II, S. 25—51. Leipzig, 1836. Both these articles may be read with pleasure and instruction even today.

(13) C. F. GAUSS: "Theoria Interpolationis methodo nova tractata," *Werke*, Band III, S. 265—327. Göttingen, 1866. (This is from *Nachlass*, but Gauss was occupied with it in 1805.) See also J. F. Encke, "Ueber Interpolation" (according to a lecture of Gauss, 1812), *Berliner Astronomisches Jahrbuch*, 1830, S. 265. Gauss' formula with numerical values of coefficients is given by Bauschinger, No. 38. See our No. 17.

(14) A. L. CAUCHY: *Cours d'analyse de l'École polytechnique.* 1st Partie, Note v, pp. 525—530, "Sur la Formule de Lagrange relative à l'Interpolation." Paris, 1821. Cauchy proposes a formula involving the *ratio* of two expressions of different orders each similar to the Lagrangian. I am not aware that it has ever been adopted in practice, but it looks as if it might be of distinct value for asymptotic forms, though for cases even of six entries I find it may be very lengthy.

(15) BESSEL, F. W.: "Anleitung und Tafeln die stündliche Bewegung des Mondes zu finden," *Astronomische Nachrichten*, Bd. II, No. 33, S. 139—143. Altona, 1824 (May, 1823). Bessel draws attention to the use of the forward difference formula (as in the *Connoisance des Temps*) and suggests use of backward differences as well. He then deduces the formula now known as "Bessel's Formula" and remarks that "ich mich nicht erinnere sie irgendwo gefunden zu haben." It is Newton's *Casus II* somewhat complicated by choosing the end instead of the middle of the interval for origin of time. It has passed into the textbooks as *Bessel's Formula.* I think Bessel himself recognised this in 1842, though he does not directly admit it, but he then speaks of Newton's formulae as the most advantageous interpolation formulae. On S. 142 Bessel gives another formula which appears again to be Newton's *Casus II* arranged according to powers of Newton's x (Bessel's t) and with a better, but far from perfect difference notation.

(16) F. W. BESSEL: "Analyse der Finsternisse. Abhandlung x, *Astronomische Untersuchungen.* Königsberg, 1842. In the preface Bessel says that *Abschnitt II* is wholly new. On S. 150—151 of this section Bessel gives two interpolation formulae—those usually called after Stirling—but here rightly attributed to Newton. Of these he writes: "Die vortheilhaftsten Interpolationsformeln—die Newtonschen—sind bekanntlich die folgenden..." (S. 150). Thus in 1842 Bessel recognised the Newtonian formulae to be the most advantageous. It is impossible to doubt that between 1823 and 1842 Bessel had discovered that his formula was Newton's. The matter is of such importance—the vindication of Newton's claim to three fundamental interpolation formulae—that I feel bound to reproduce Bessel's exhibition of Newton's formulae:

I = Newton's *Casus II* (see No. 5, p. 97)

Argument	Function	Δ_1	Δ_2	Δ_3	Δ_4
$-\frac{1}{2}$	a_1		c_1		e_1
		b		d	
$+\frac{1}{2}$	a'		c'		e'

etc.,

$$a=\tfrac{1}{2}(a_1+a'),\quad c=\tfrac{1}{2}(c_1+c'),\quad e=\tfrac{1}{2}(e_1+e'),\ \text{etc.,}$$

$$f(t)=a+tb+\frac{t^2-\frac{1}{4}}{2}c+\frac{t(t^2-\frac{1}{4})}{2\,.\,3}d+\frac{(t^2-\frac{1}{4})(t^2-\frac{9}{4})}{2\,.\,3\,.\,4}e+\frac{t(t^2-\frac{1}{4})(t^2-\frac{9}{4})}{2\,.\,3\,.\,4\,.\,5}f+\text{etc.}$$

This agrees also with Stirling's *Casus secundus*, except that Stirling makes his interval two, instead of unity (see No. 5^{bis}, p. 106), i.e. Newton's x = Bessel's t (in above) = Stirling's $\frac{1}{2}z$. This is a mid-panel formula with origin at centre of the panel. Transfer the origin to the end of the panel by putting $t=n-\frac{1}{2}$ and we have the formula given by Bessel in 1823, and called after him in the text-books: see Nos. 15 and 38.

II = Newton's *Casus I* (see No. 5, p. 96):

Argument	Function	Δ_1	Δ_2	Δ_3	Δ_4
		b_1		d_1	
0	a		c		e
		b'		d'	

etc.,

$$2b=(b_1+b'),\quad 2d=(d_1+d'),\quad 2f=(f_1+f'),\ \text{etc.,}$$

$$f(t)=a+tb+\frac{t^2}{2}c+\frac{t(t^2-1)}{2\,.\,3}d+\frac{t^2(t^2-1)}{2\,.\,3\,.\,4}e+\frac{t(t^2-1)(t^2-4)}{2\,.\,3\,.\,4\,.\,5}f+\text{etc.}$$

This agrees also with Stirling's *Casus I* (No. 5^{bis}, p. 104) with changes of difference notation. It is called after Stirling in the text-books—erroneously. Bessel says that Newton's formulae should be used for arguments from $-\frac{1}{2}$ to $+\frac{1}{2}$. I think this is only partially accurate. (I) should be used for the mid-half of a panel, i.e. $-\frac{1}{4}$ to $\frac{1}{4}$. Beyond this on either side (II) should be used, i.e. the limits are $-\frac{1}{4}$ to $\frac{1}{4}$ again. The proper range for the "Bessel formula" of the text-books is $n=\frac{1}{4}$ to $n=\frac{3}{4}$, and for the "Stirling's formula" of the text-books $-\frac{1}{4}$ to $\frac{1}{4}$. Newton's I and II are *not* competing formulae, but are to be used for different parts of the panel, i.e. Stirling's formula is a mid-point and Bessel's formula a mid-panel formula. Neither formula should bear the name of Stirling or Bessel, any more than the formula due to Waring should be called after Lagrange. Stirling himself in his *Scholion* to *Propositis* xx (No. 5^{bis}, p. 107) gives the credit to Newton.

(17) REMARK. If we take Newton's *Casus I*, and *destroy its symmetry* by writing

$$b = \tfrac{1}{2}(b_1 + b') = b' - \tfrac{1}{2}(b' - b_1) = b' - \tfrac{1}{2}c,$$
$$d = \tfrac{1}{2}(d_1 + d') = d' - \tfrac{1}{2}(d' - d_1) = d' - \tfrac{1}{2}e, \text{ etc.,}$$

we obtain on collecting terms,

$$f(t) = a + tb' + \frac{t(t-1)}{1.2}c + \frac{t(t^2-1)}{1.2.3}d' + \frac{t(t^2-1)(t-2)}{1.2.3.4}e + \dots,$$

which is the so-called Gauss' formula of the text-books. Thus Gauss' formula is only the slightest transformation of Stirling's or Newton's *Casus I*. It is difficult to see what is gained by introducing this asymmetry as b_1, d_1, etc. must be known, otherwise c, e, etc. could not be found. See Nos. 13 and 38.

(18) LEGENDRE: *Traité des Fonctions elliptiques...*, Tome II, Chapitre XV, pp. 201—207. Paris, 1826. Gives the forward difference formula for two variates and illustrates its use amply.

(19) F. BRÜNNOW: *Lehrbuch der sphärischen Astronomie.* Berlin, 1831. "Die Interpolationsrechnung," S. 24—42. Of no present importance. Formulae S. 30—31, appear to be Gauss, S. 32 to be Bessel, but no names or references, except to Encke's papers.

(20) A. L. CAUCHY: "Mémoire sur l'Interpolation," *Journal de mathématiques pures et appliquées*, Tome II, pp. 193—205. Paris, 1837. A novel interpolation method of an approximational character, entirely different from that of parabolic curves. I am not at all convinced of the value of the method, either for the case of an n-constant curve run through n points, or in the case of the same curve through m-points ($m > n$) supposed subject to errors.

(21) A. DE MORGAN: *The Differential and Integral Calculus*, Chapter XVIII, "On Interpolation and Summation," pp. 542—560. London, 1842. As usual a certain amount of material, not readily found elsewhere.

(22) E. BRASSINNE: "Sur l'Interpolation" (an extension of Cauchy's method in the *Cours d'analyse*, see No. 14), *Journal de mathématiques pures et appliquées*, Tome XI, pp. 177—183. Paris, 1846. The paper indicates the variety of formulae which are included under Cauchy's "ratio" formula. See No. 14.

(23) J. BIENAYMÉ: "Sur les différences qui distinguent l'interpolation de M. Cauchy de la methode des moindres carrés, et qui assurent la supériorité de cette methode," *Journal de mathématiques pures et appliquées*, Tome XVIII, pp. 299—308. Paris, 1853. (Criticism of Cauchy's method of 1837.) See also *Comptes rendus*, Tome XXXVII, pp. 5—13, and Cauchy's various memoirs in the same volume. Notwithstanding Bienaymé's criticism, the method seems approved by Bauschinger (No. 42, S. 817). Rice (No. 34, p. 233) cites the literature in his very brief bibliography, but I have not been able to trace any reference to the method in his text.

(24) P. A. Hansen: *Tables de la Lune*, pp. 68—71. Londres, 1857. Gives a table as if for the first time of the coefficients of 3rd, 4th, and 5th difference terms in Stirling's 1st Difference Formula, i.e. Newton's *Casus I.* These have been copied generally into the books of mathematical tables, e.g. No. 38*.

(25) P. Tchebycheff: "Sur l'Interpolation dans le cas d'un grand nombre de données fournies, par les observations," *Academie Impériale des Sciences, Mémoires*, VII°, série 1, 1859, No. 5. St Petersburg, 1859. Explanatory and illustrative memoirs are O. Backlund, "Ueber die Answendung einer von P. Tchebycheff vorgeschlagenen Interpolationsmethode," *Bulletin de Academie des Sciences*, Tome XXIX, pp. 477—498. St Petersburg, 1884, and P. Harzer, "Ueber eine von Herrn Tchebycheff augegebene Interpolationsformal," *Astronomische Nachrichten*, Bd. CXV, S. 337—384. Kiel, 1886. Gives tables to facilitate use.

(26) W. S. B. Woolhouse: *On Interpolation, Summation and the Adjustment of Numerical Tables.* London, 1865. See also *The Assurance Magazine* and *Journal of the Institute of Actuaries*, Vol. XI, pp. 61—88, 301—332; Vol. XII, pp. 136—176. London, 1864—1866. Largely occupied with an individual formula in a variety of forms and its application to quadrature. (The *Casus primus* of Stirling (Newton) expressed only slightly differently is Woolhouse's (α) (p. 8 of the offprint).)

(27) J. L. Lagrange: "Sur les Interpolations (1778)," *Oeuvres*, Tome VII, pp. 535 *et seq.* Paris, 1870. Also "Mémoire sur la Méthode d'Interpolation (1792—3)," *Oeuvres*, Tome V, pp. 663—684. Paris, 1870. The empirical method discovered by Briggs and followed by Mouton receives in these papers its mathematical statement. The former paper has a useful historical summary.

(28) G. Boole (Ed. Moulton): *A Treatise on the Calculus of Finite Differences*, Chapter III, "On Interpolation and Mechanical Quadrature," pp. 33—61. London, 1872. May still be of some value to the English student.

(29) W. Chauvenet: *A Manual of Spherical and Practical Astronomy. Interpolation by Differences of any Order*, Vol. I, pp. 79—91. Philadelphia, 1874. Supplies numerical illustrations from astronomy of some of the chief interpolation formulae. Like all the text-books on astronomy I know, it cites Bessel's formula, but omits to state where and when it was first published.

* The first appearance of tabled coefficients of Bessel's formula are to be found in a work by Sawitch, but W. C. Goetze in his translation of Sawitch: *Abriss der praktischen Astronomie*, Hamburg, 1850 (see preface and S. 425, 434) states that the originals are in Wrangel's *Hülfstafeln*, which I cannot find. C. F. W. Peters in his *Astronomische Tafeln und Formeln*, Hamburg, 1871, gives S. 103 the coefficients of Bessel's formula and S. 104—5 those of Stirling's formula. The former differ from those in Sawitch (? Wrangel) in that they proceed by decimals of the panel, and not to minutes of the hour or day. Peters' values appear to be those copied in later books, often without acknowledgment.

(30) C. W. Merrifield: "Report on the present State of Knowledge of the Application of Quadratures and Interpolation to Actual Data," *British Association Report* (Swansea, 1880), pp. 321—378. London, 1880. Useful, but not as full as one might have wished; this is especially true in regard to bibliographical details.

(31) E. Heine: *Handbuch der Kugelfunctionen*, Bd. II, Anwendungen, I Theil, "Mechanische Quadratur," S. 1—31. Berlin, 1881. This is a good account of the methods of using an interpolation formula to obtain quadrature, especially the Gaussian method for selecting the best unequal intervals between ordinates to give the most accurate value of the interval. (Gauss, *Werke*, Bd. III, S. 163—196. The original memoir (1876) belongs to quadrature rather than to interpolation.)

(32) R. Radau: *Études sur les formules d'Interpolation.* Paris, 1891. See also *Bulletin Astronomique*, Tome VIII, pp. 273—, 325—, 376—, 425—. Paris, 1891. A useful paper in several ways, especially for those who desire further light on the methods of Cauchy and Tchebycheff, which are treated at rather disproportionate length. Radau in some respects antedates Sheppard.

(33) A. A. Markoff: *Differenzrechnung.* Leipzig, 1896. (Translation of the Russian edition, Petrograd, 1891.) Very full discussion and references for history and practise of Interpolation.

(34) Herbert L. Rice: *The Theory and Practise of Interpolation.* Lynn, Mass. 1899. A useful, but somewhat disappointing book.

(35) J. D. Everett: "On a Central Difference Interpolation Formula," *British Association Transactions.* Bradford, 1900, pp. 648—650. Followed by a Note stating that Newton discovered first the forward difference formula in Lemma 5 which follows Prop. 40 in Book III of the *Principia.* First statement of mid-panel central difference formula.

(36) W. F. Sheppard: "Central Difference Formulae," *Proceedings of the London Mathematical Society*, Vol. XXXI, pp. 449—488. London, 1900. Contains a great variety of central difference formulae.

(37) J. D. Everett: "On Interpolation Formulae," *The Quarterly Journal of Pure and Applied Mathematics*, Vol. XXXII, pp. 306—313. London, 1901.

(38) J. Bauschinger: *Tafeln zur theoretischen Astronomie.* Leipzig, 1901. Gives Newton's, Stirling's, Bessel's and Gauss' formulae and the corresponding differentials in terms of differences (S. 136—7) with tables of coefficient values up to fourth differences for interpolation, differentiation and quadrature—usually to five decimal places (S. 138—140). No references to original sources.

(39) George King: *Text Book of the Institute of Actuaries*, Part II, pp. 447 *et seq.* London, 1902 (pp. 434—457. London, 1887). A useful discussion of the subject, with illustrations from actuarial data.

(40) J. D. EVERETT: "On a New Interpolation Formula," *Journal of the Institute of Actuaries*, Vol. XXXV, pp. 452—458. London, 1901. Another publication of the mid-panel central difference interpolation formula.

(41) J. D. EVERETT: Article "Interpolation," *Encyclopaedia Britannica*, 10th Edition, Vol. XXIX. London, 1902, pp. 540—542. An article not wholly superseded by the more recent account in the 11th edition.

(42) J. BAUSCHINGER: "Interpolation," *Encyclopädie der mathematischen Wissenschaften*. Erster Band, Zweiter Teil, S. 799—820. Leipzig, 1902—4. Useful, but somewhat brief.

(43) H. BRUNS: *Grundlinien des wissenschaftlichen Rechnens*. Abschnitte I, II and IX, S. 11—61, 149—159. Leipzig, 1903. Not very adequate. Wanting on the side of practical applications and illustrations.

(44) W. F. SHEPPARD: "On the Accuracy of Interpolation by Finite Differences," *Proceedings of the London Mathematical Society*, Series 2, Vol. IV, pp. 320—341. London, 1906. Deals with the important points of error in interpolation due (a) to the possibility of a difference of something under a half unit in the last figure retained in the interpolants, and (b) to errors of observation when the interpolants are measured quantities.

(45) W. F. SHEPPARD: Article "Interpolation," *Encyclopaedia Britannica*, 11th Edition, Vol. XIV, pp. 706—710. Cambridge, 1910. Most complete English account of the present state of our knowledge, limited by the conditions of publication on the bibliographical side.

(46) DAVID GIBB: "A Course in Interpolation and Numerical Integration." *Edinburgh Mathematical Tracts*, No. 2. London, 1915. A helpful book for the learner; it does not, however, cover the ground of central difference methods, nor the bivariate tables of our tracts.

(47) S. A. JOFFE: "Interpolation-Formulae and Central Difference Notation," *Actuarial Society of America, Transactions*, Vol. XVIII, pp. 72—98. New York, 1917. I regret that I did not know of this paper until the present bibliography was in proof. Joffe does justice to Newton, but apparently had not come across Waring. I do not think his terminology and in particular his symbols are at all likely to be generally adopted, but he had studied the history of the subject far more closely than previous writers.

(48) DUNCAN C. FRASER: "Newton's Interpolation Formulas," *Journal of the Institute of Actuaries*, Vol. LI, pp. 77—106, 211—232. London, 1918—1919. Also as offprint (Layton Brothers). This paper reproduces all Newton's work on Interpolation, provides a facsimile of the *Methodus differentialis* and adds letters referring to Newton's views, translations and notes.

(49) A. KOWALEWSKI: *Newton, Cotes, Gauss, Jacobi.* Vier grundlegende Abhandlungen über Interpolation und genäherte Quadratur (1711, 1722, 1814, 1826). Übersetzt bzw. herausgegeben und mit einem erläuternden Anhang versehn. (Veit.) 1917. Nothing of special note for English readers as to Newton and Cotes, but a good account of Gauss' (No. 13) and Jacobi's (*Crelle's Journal*, Band I, S. 301, Berlin, 1826 and Band II, S. 223, Berlin, 1827) contributions to quadrature by interpolation formulae for unequal intervals is provided.

(50) J. P. GRAM: *Om Raekkeudviklinger, bestemte ved Hjaelp af de mindste Kvadraters Methode.* Kjøbenhavn, 1879. See also *Crelle's Journal*, Band XCIV, S. 41—73. Berlin, 1883. A valuable paper on generalised interpolation, i.e. interpolation by series of functions other than integer powers of x.

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TRACTS FOR COMPUTERS

EDITED BY KARL PEARSON, F.R.S.

No. III

On the Construction of Tables and on Interpolation.
Part II. Bi-variate Tables

BY KARL PEARSON, F.R.S.

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No. III

TO MY FELLOW MEMBERS OF THE ACTUARIES' CLUB

"Sir, I beseech you to accept or pardon these trifling interpolations, which I have presumed to send you." *Letter of John Evelyn, F.R.S. to John Aubrey, F.R.S.*, 1675.

CONTENTS

ON THE CONSTRUCTION OF TABLES AND ON INTERPOLATION

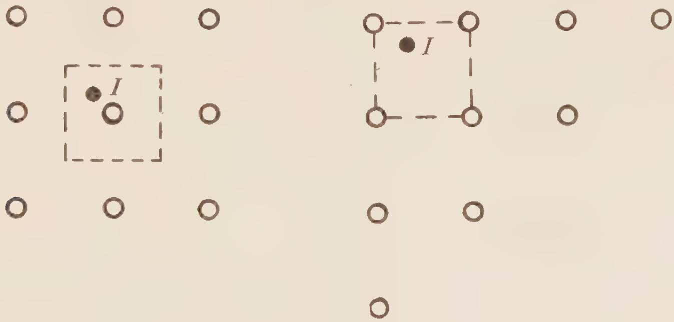
PART II. BI-VARIATE INTERPOLATION

By Karl Pearson, F.R.S.

General Remarks on Bi-variate Interpolation.

In this second part of our investigation we lose the guidance and suggestion hitherto provided by the many writers on uni-variate interpolation. The practical need for dealing with multi-variate tables seems to be a recent one and most writers on interpolation, including those on finite differences, appear to content themselves with a brief reference to the use of *forward* differences in bi-variate interpolation. We have already pointed out the defects of forward differences even in uni-variate interpolation and argued that they must be replaced by central differences, i.e. if we are working from n values, these n values should be taken from the nearest n values to the interpolate, and this is far from the case with forward differences. The argument is greatly strengthened when we come to multiple interpolation. The nine nearest interpolants to a given interpolate are by no means nine out of the ten points involved in using the cubic differences of a forward difference bi-variate formula. It is more than reasonable to suggest that the nine nearest interpolants will give a better value for the interpolate than the ten forward difference interpolants which leave the interpolate on the *edge* of the interpolant group. The following diagrams illustrate this difference:

(i) (ii)



I = plan of interpolate, o = plans of interpolants. The dotted boundary indicates the range of interpolate corresponding to each system.

These schemes, however, show in its most elementary form a special difficulty of multi-variate interpolation. Geometrically the forward difference formula is only the process of throwing a surface of the form

$$z = \text{function of integer ascending powers of } x \text{ and } y$$

through ten selected points. Such a surface is an obvious cubic surface.

But if we take the *nine* nearest points as in (i) what is to be the form of the interpolant surface? We can only obtain it by leaving out one term of the general cubic surface, and if we are to retain the symmetry of our formula, the one term possible must be xy. We start accordingly with the surface

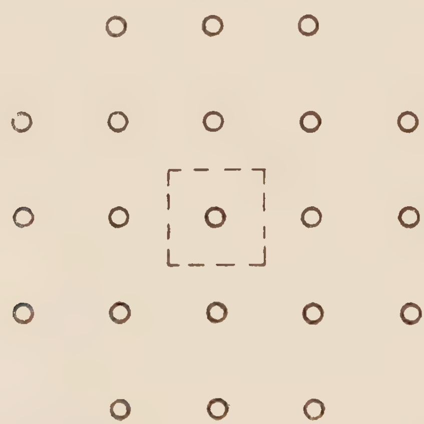
$$z = c_0 + c_1 x + c_2 y + c_3 x^2 + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 x y^2 + c_9 y^3,$$

and find it cannot be passed through nine arbitrary points whose plans are a rectangular network without hopeless restrictions on the corresponding z's. For example

$$z_{1,1} - z_{-1,1} = z_{1,-1} - z_{-1,-1},$$

and if such a relation holds, our constants are not uniquely determinable. This type of difficulty pursues us throughout bi-variate interpolation. If we take the 21 nearest points as in this diagram,

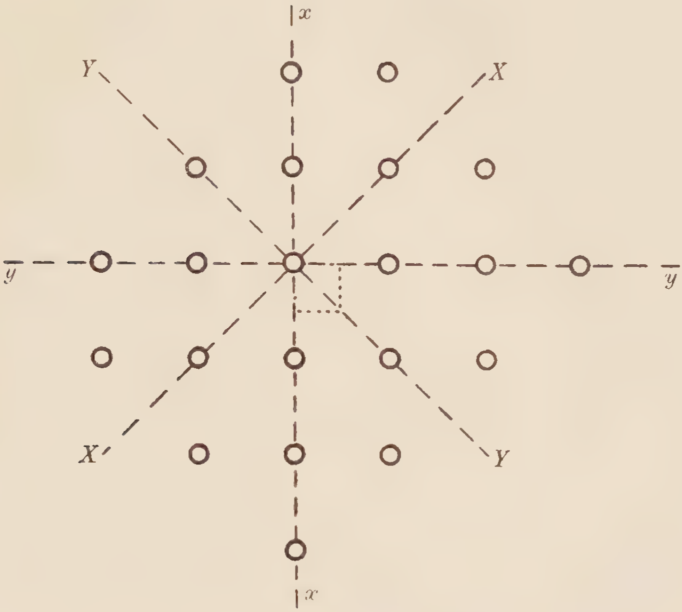
(iii)



their plans forming a rectangular network, but the general quintic with 21 constants cannot be determined (for reasons similar to those just stated) for

the above 21 interpolants. A quintic surface can however be found for the 21 following points*:

(iv)



But these are not the 21 nearest interpolants to any point in the square area, the points are not symmetrical in the rows and columns of the table, i.e. parallel to xx and yy, but only with regard to the diagonal YY. There seems therefore something artificial about the arrangement.

The only course seems to be to obtain a surface passing through the system by dropping some of the lower order terms and introducing some of the higher into the equation of the surface. What terms are to be supplied will usually be suggested by expanding the uni-variate interpolation into bi-variate interpolation in the manner we have presently to indicate.

A full mathematical treatment of the difficulties—and the interesting problems as to surfaces through point systems—raised by multi-variate interpolation, I have not succeeded in finding. As far as I have been able to discover the discussion of multi-variate interpolation has been confined to:

* I owe this interesting system to my colleague, Mr H. E. Soper, who has expressed the result in differences.

(i) A suggestion of the use of the bi-variate forward difference formula.

The great objection to this formula is that it does not weight equally all interpolants which have the same degree of proximity to the interpolate. This objection is the objection to the forward difference formula in uni-variate interpolation but intensified.

(ii) A suggestion that we should interpolate values first for one variate, say x, and then from the resulting values interpolate for the other, y.

Unfortunately if we have to use high differences we must use our interpolation formula a high number of times to obtain the requisite number of values of y for a given x, before we again use the interpolation formula to obtain the required value of y.

The method is therefore not only laborious in the case of high moments, but it will be found that if we interpolate for y first and then for x the results will not necessarily be in accordance*.

(iii) A method due to Mr W. Palin Elderton. In 1902 he suggested a process (*Biometrika*, Vol. II, p. 105) of practically working interpolation on two variate tables, which is akin to (ii) but uses function values instead of differences. He has further given an ingenious process for uni-variate interpolation on a bi-variate table (*Biometrika*, Vol. VI, p. 94, 1908), but I venture to think this might present certain difficulties when applied to some of the 'precipitous' tables with which we have recently had to deal.

In dealing with the double entry $G(r, \nu)$ table of the *British Association Report of* 1899 I remarked on the difficulty of giving equal weight to all adjacent points, and wrote:

"With regard to interpolation formulae for Tables of double entry, we have been unable to discover much consideration of the subject, possibly because hitherto such tables have been rather rare. We do not know of any formulae, similar to those for interpolation on a curve for interpolating on surfaces" (p. 68).

When writing this I had in view something of the nature of Lagrange's formula, but for two dimensions. I did not perceive at that time, nor for a long time afterwards, when dealing with tables where it was requisite to use high differences for interpolation purposes, that the central difference formula extended to two dimensions was essentially what I needed.

* The reader can test this at once in a simple case by using linear differences. Accordance depends on a plane passing through four points, which as a rule it cannot do.

General Mid-panel Central Difference Formula.

Consider the diagram (v) on p. 9, and let the function value to be interpolated be in the square formed by the tabled values $z_{0,0}$, $\dot{z}_{0,1}$, $z_{1,1}$ and $z_{1,0}$. The required value is $z_{\theta,\chi}$ dividing the horizontal unit in the ratio χ to ψ and the vertical unit in the ratio θ to ϕ. Taking the tabled values as indicated in the accompanying figure, the four 'nearest' values may be considered as $z_{0,0}$, $z_{0,1}$, $z_{1,1}$. $z_{1,0}$: if we go a stage further we must introduce the eight values $z_{-1,1}$, $z_{0,2}$, $z_{1,2}$, $z_{2,1}$, $z_{2,0}$, $z_{1,-1}$, $z_{0,-1}$, $z_{-1,0}$, lying on the broken line hexagon of the figure. The use of these twelve points corresponds to using second central differences. If we now take the twelve points on the dotted line hexagon, i.e. $z_{-2,1}$, $z_{-1,2}$, $z_{0,3}$, $z_{1,3}$, $z_{2,2}$, $z_{3,1}$, $z_{3,0}$, $z_{2,-1}$, $z_{1,-2}$, $z_{0,-2}$, $z_{-1,-1}$, $z_{2,0}$, we shall base our interpolation on the 24 'nearest' points. To obtain our interpolation formulae in a Lagrange form we should need therefore to determine surfaces, whose constants are fixed by 4, 12, 24 points respectively. It is not easy, however, to choose such surfaces *a priori*; they cannot be selected like the high order parabolae of uni-variate interpolation. Four points are too few to fix a general quadric surface of the type

$$z = a_0 + b_1x + c_1y + \tfrac{1}{2}(d_1x^2 + 2e_1xy + f_1y^2) + \text{etc.}$$

and twelve points for a similar type of cubic surface are too many. In the same way 24 points are too many for the bi-quadratic surface of the like form and a sextic requires 28 points to fix it. It is only when we have obtained the central difference formula for two variates that we can see light on the nature of the required surfaces by replacing the differences by the tabled functions, i.e. z's.

Let the interpolate divide the square formed by the four nearest interpolants in the x-ratio $\theta : \phi$ $(\phi = 1 - \theta)$ and the y-ratio $\chi : \psi$ $(\psi = 1 - \chi)$. Then the usual mid-panel central difference formula provided us with

$$\begin{aligned} z_{\theta,\chi} = \phi z_{0,\chi} + \theta z_{1,\chi} &- \frac{1}{6}\theta\phi\{(1+\phi)\,\delta^2 z_{0,\chi} + (1+\theta)\,\delta^2 z_{1,\chi}\} \\ &+ \frac{1}{120}\theta(1+\theta)\,\phi(1+\phi)\{(2+\phi)\,\delta^4 z_{0,\chi} + (2+\theta)\,\delta^4 z_{1,\chi}\} \\ &- \frac{1}{7\,!}\theta(1+\theta)(2+\theta)\,\phi(1+\phi)(2+\phi)\{(3+\phi)\,\delta^6 z_{0,\chi} + (3+\theta)\,\delta^6 z_{1,\chi}\} \\ &- \text{etc.} \;\ldots\ldots\text{(i)}. \end{aligned}$$

If we stop at the first line, this is a cubic passing through the four points on the χ-line; if we stop at the second line, this is a quintic passing through

the six points on the χ-line, and if we stop at the third line we have a septenic passing through the eight points on the χ-line. Consequently if we expand $z_{0,\chi}$ and $z_{1,\chi}$ exactly as we have expanded $z_{\theta,\chi}$ we shall obtain surfaces of the sixth, tenth and fourteenth orders passing respectively through 16, 36 and 64 points arranged in squares of 4×4, 6×6 and 8×8 points. But the resulting formulae contain respectively fourth, eighth and twelfth order differences, and we do not propose to keep in the several cases terms higher than the second, fourth and sixth order differences, so that the formulae shall be correct up to but not including fourth, sixth and eighth order difference terms respectively. Looked at from the Lagrangian standpoint, what points will our surface pass through under these conditions and what will be its order?

We have, if δ' denote a y-variate central difference operator,

$$\begin{aligned}
z_{0,\chi} &= \psi z_{0,0} + \chi z_{0,1} - \frac{1}{6}\chi\psi\{(1+\psi)\,\delta'^2 z_{0,0} + (1+\chi)\,\delta'^2 z_{0,1}\} \\
&\quad \frac{1}{5!}\chi(1+\chi)\,\psi(1+\psi)\{(2+\psi)\,\delta'^4 z_{0,0} + (2+\chi)\,\delta'^4 z_{0,1}\} \\
&\quad -\frac{1}{7!}\chi(1+\chi)(2+\chi)\,\psi(1+\psi)(2+\psi)\{(3+\psi)\,\delta'^6 z_{0,0} + (3+\chi)\,\delta'^6 z_{0,1}\} + \dots
\end{aligned} \qquad \text{......(ii)},$$

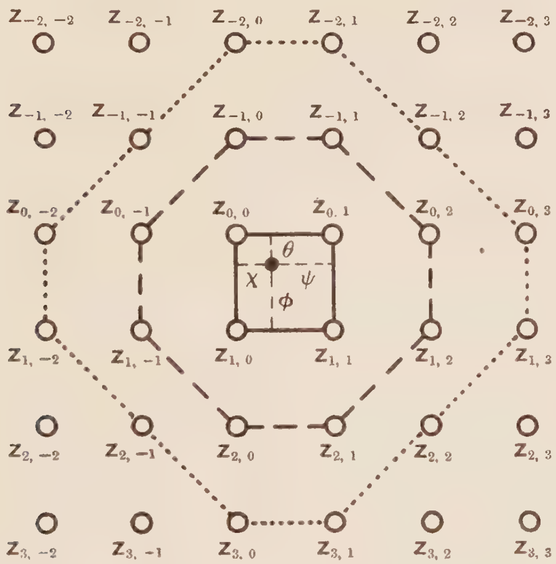
$$\begin{aligned}
z_{1,\chi} &= \psi z_{1,0} + \chi z_{1,1} - \frac{1}{6}\chi\psi\{(1+\psi)\,\delta'^2 z_{1,0} + (1+\chi)\,\delta'^2 z_{1,1}\} \\
&\quad +\frac{1}{5!}\chi(1+\chi)\,\psi(1+\psi)\{(2+\psi)\,\delta'^4 z_{1,0} + (2+\chi)\,\delta'^4 z_{1,1}\} \\
&\quad -\frac{1}{7!}\chi(1+\chi)(2+\chi)\,\psi(1+\psi)(2+\psi)\{(3+\psi)\,\delta'^6 z_{1,0} + (3+\chi)\,\delta'^6 z_{1,1}\} + \dots
\end{aligned} \qquad \text{......(iii)}.$$

We have then to operate with δ^2 and δ^4 on these two expressions and substitute in the above formula for $z_{\theta,\chi}$. For most practical work, however, we can hardly with advantage retain terms beyond the fourth order differences. As we shall see the formula then denotes a sextic, but it only passes through the 24 nearest points, whereas the general sextic could theoretically be made to pass through 28 points. Actually what we have obtained is an approximation to the tenth order surface which passes through a 6×6 square of points, the approximation consisting in the neglect of higher order differences. The following is the result reached:

$$\begin{aligned}
z_{\theta,\chi} &= \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\
&\quad - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\
&\quad - \tfrac{1}{6}\chi\psi\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}
\end{aligned}$$

$$+\tfrac{1}{120}\theta\phi(1+\theta)(1+\phi)\{(2+\phi)(\psi\delta^4 z_{0,0}+\chi\delta^4 z_{0,1})$$
$$+(2+\theta)(\psi\delta^4 z_{1,0}+\chi\delta^4 z_{1,1})\}$$
$$+\tfrac{1}{36}\theta\phi\chi\psi\{(1+\phi)(1+\psi)\delta^2\delta'^2 z_{0,0}+(1+\phi)(1+\chi)\delta^2\delta'^2 z_{0,1}$$
$$+(1+\theta)(1+\psi)\delta^2\delta'^2 z_{1,0}+(1+\theta)(1+\chi)\delta^2\delta'^2 z_{1,1}\}$$
$$+\tfrac{1}{120}\chi\psi(1+\chi)(1+\psi)\{(2+\psi)(\phi\delta'^4 z_{0,0}+\theta\delta'^4 z_{1,0})$$
$$+(2+\chi)(\phi\delta'^4 z_{0,1}+\theta\delta'^4 z_{1,1})\}$$
$$-\tfrac{1}{720}\theta\phi\chi\psi(1+\theta)(1+\phi)\{(2+\phi)[(1+\psi)\delta^4\delta'^2 z_{0,0}+(1+\chi)\delta^4\delta'^2 z_{0,1}]$$
$$+(2+\theta)[(1+\psi)\delta^4\delta'^2 z_{1,0}+(1+\chi)\delta^4\delta'^2 z_{1,1}]\}$$
$$-\tfrac{1}{720}\theta\phi\chi\psi(1+\chi)(1+\psi)\{(2+\psi)[(1+\phi)\delta^2\delta'^4 z_{0,0}+(1+\theta)\delta^2\delta'^4 z_{1,0}]$$
$$+(2+\chi)[(1+\phi)\delta^2\delta'^4 z_{0,1}+(1+\theta)\delta^2\delta'^4 z_{1,1}]\}$$
$$-\tfrac{1}{5040}\theta\phi(1+\phi)(2+\phi)(1+\theta)(2+\theta)[(3+\phi)(\psi\delta^6 z_{0,0}+\chi\delta^6 z_{0,1})$$
$$+(3+\theta)(\psi\delta^6 z_{1,0}+\chi\delta^6 z_{1,1})]$$
$$-\tfrac{1}{5040}\psi\chi(1+\psi)(2+\psi)(1+\chi)(2+\chi)[(3+\psi)(\phi\delta'^6 z_{0,0}+\theta\delta'^6 z_{1,0})$$
$$+(3+\chi)(\phi\delta'^6 z_{0,1}+\theta\delta'^6 z_{1,1})]$$
$$+\text{terms of 8th order} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv).}$$

(v)



Here

$$\delta^2 z_{0,0} = z_{1,0} - 2z_{0,0} + z_{-1,0}, \qquad \delta'^2 z_{0,0} = z_{0,1} - 2z_{0,0} + z_{0,-1},$$
$$\delta^2 z_{0,1} = z_{1,1} - 2z_{0,1} + z_{-1,1}, \qquad \delta'^2 z_{0,1} = z_{0,2} - 2z_{0,1} + z_{0,0},$$
$$\delta^2 z_{1,0} = z_{2,0} - 2z_{1,0} + z_{0,0}, \qquad \delta'^2 z_{1,0} = z_{1,1} - 2z_{1,0} + z_{1,-1},$$
$$\delta^2 z_{1,1} = z_{2,1} - 2z_{1,1} + z_{0,1}, \qquad \delta'^2 z_{1,1} = z_{1,2} - 2z_{1,1} + z_{1,0},$$
$$\delta^4 z_{0,0} = \delta^2 z_{1,0} - 2\delta^2 z_{0,0} + \delta^2 z_{-1,0}, \quad \delta'^4 z_{0,0} = \delta'^2 z_{0,1} - 2\delta'^2 z_{0,0} + \delta'^2 z_{0,-1},$$
$$\delta^4 z_{0,1} = \delta^2 z_{1,1} - 2\delta^2 z_{0,1} + \delta^2 z_{-1,1}, \quad \delta'^4 z_{0,1} = \delta'^2 z_{0,2} - 2\delta'^2 z_{0,1} + \delta'^2 z_{0,0},$$
$$\delta^4 z_{1,0} = \delta^2 z_{2,0} - 2\delta^2 z_{1,0} + \delta^2 z_{0,0}, \quad \delta'^4 z_{1,0} = \delta'^2 z_{1,1} - 2\delta'^2 z_{1,0} + \delta'^2 z_{1,-1},$$
$$\delta^4 z_{1,1} = \delta^2 z_{2,1} - 2\delta^2 z_{1,1} + \delta^2 z_{0,1}, \quad \delta'^4 z_{1,1} = \delta'^2 z_{1,2} - 2\delta'^2 z_{1,1} + \delta'^2 z_{1,0},$$
$$\delta^2\delta'^2 z_{0,0} = \delta'^2 z_{1,0} - 2\delta'^2 z_{0,0} + \delta'^2 z_{-1,0} = \delta^2 z_{0,1} - 2\delta^2 z_{1,0} + \delta^2 z_{0,-1},$$
$$\delta^2\delta'^2 z_{0,1} = \delta'^2 z_{1,1} - 2\delta'^2 z_{0,1} + \delta'^2 z_{-1,1} = \delta^2 z_{0,2} - 2\delta^2 z_{0,1} + \delta^2 z_{0,0},$$
$$\delta^2\delta'^2 z_{1,0} = \delta'^2 z_{2,0} - 2\delta'^2 z_{1,0} + \delta'^2 z_{0,0} = \delta^2 z_{1,1} - 2\delta^2 z_{1,0} + \delta^2 z_{1,-1},$$
$$\delta^2\delta'^2 z_{1,1} = \delta'^2 z_{2,1} - 2\delta'^2 z_{1,1} + \delta'^2 z_{0,1} = \delta^2 z_{1,2} - 2\delta^2 z_{1,1} + \delta^2 z_{1,0}. \quad \text{.........(v).}$$

Thus to obtain interpolation easily and correctly up to but not including the sixth central differences, each tabulated entry should be accompanied by five differences for that value, namely: δ^2, δ'^2, δ^4, δ'^4, and $\delta^2\delta'^2$. If space does not permit of printing all five, then δ^2 and δ'^2 should at least be given. From them δ^4, δ'^4, and $\delta^2\delta'^2$ are easily deduced by aid of the above relations.

The proposal I wish to make is that in future with machine instead of logarithmic calculation the tabling of first differences is of no importance, that for a uni-variate table second and fourth central differences should be tabled as in the Tables of the Digamma and Trigamma Functions (Cambridge Press) recently issued by this Laboratory. In many cases it will be sufficient to table δ^2, which will give accuracy to the third difference order and is therefore far better than simply tabling the first difference. In tables of double entry there should be, as in Vega's 10-figure logarithm tables, at least one separate facing page of differences and it should record, where feasible, the five fundamental central differences. It is the four sets of five differences at the angles of the nearest square which are requisite for the full central difference formula.

In the uni-variate central difference formula

$$\begin{aligned} z_0 = \phi z_0 + \theta z_1 - \tfrac{1}{6}\theta\phi\,\{(1+\phi)\,\delta^2 z_0 + (1+\theta)\,\delta^2 z_1\} \\ + \tfrac{1}{120}\theta\phi\,(1+\theta)(1+\phi)\,\{(2+\phi)\,\delta^4 z_0 + (2+\theta)\,\delta^4 z_1\} \\ - \tfrac{1}{5040}\theta\phi\,(1+\theta)(2+\theta)(1+\phi)(2+\phi)\,\{(3+\phi)\,\delta^4 z_0 + (3+\theta)\,\delta^4 z_1\} \\ + \text{etc. } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi),} \end{aligned}$$

the coefficients of the central differences are functions of θ (since $\phi = 1 - \theta$). It is clear that a tabulation of these coefficients, i.e.

$$\tfrac{1}{6}\theta\phi(1+\phi), \quad \tfrac{1}{120}\theta\phi(1+\phi)(2+\phi),$$

and

$$\tfrac{1}{5040}\theta\phi(1+\theta)(2+\theta)(1+\phi)(2+\phi)(3+\phi),$$

would be extremely useful. For the second set of coefficients we have only to enter with ϕ instead of θ. Further it will be seen that in the bi-variate interpolation formula the coefficients which occur are the same coefficients or products of them. A table of these coefficients will therefore serve for single and double interpolation. Such a table is now in hand and will shortly be published. It should save much labour in central difference interpolation.

Illustration. The figures on the following page give 36 entries from a table of double entry. The fundamental differences are provided as far as they concern our present purpose, which is to interpolate a value in the $z_{0,0}$, $z_{0,1}$, $z_{1,1}$, $z_{1,0}$ area. To obtain nearly the highest influence of the fourth order differences we will find the value of $z_{\frac{1}{2},\frac{1}{2}}$, or the function value for $x = 3{\cdot}7$, $y = 5{\cdot}1$.

In this case our central difference formula reduces to

$$\begin{aligned} z_{\frac{1}{2},\frac{1}{2}} = {} & \tfrac{1}{4}(z_{0,0} + z_{0,1} + z_{1,1} + z_{1,0}) - \tfrac{1}{32}(\delta^2 z_{0,0} + \delta^2 z_{0,1} + \delta^2 z_{1,1} + \delta^2 z_{1,0}) \\ & - \tfrac{1}{32}(\delta'^2 z_{0,0} + \delta'^2 z_{0,1} + \delta'^2 z_{1,1} + \delta'^2 z_{1,0}) + \tfrac{3}{512}(\delta^4 z_{0,0} + \delta^4 z_{0,1} + \delta^4 z_{1,1} + \delta^4 z_{1,0}) \\ & + \tfrac{1}{256}(\delta^2\delta'^2 z_{0,0} + \delta^2\delta'^2 z_{0,1} + \delta^2\delta'^2 z_{1,1} + \delta^2\delta'^2 z_{1,0}) \\ & + \tfrac{3}{512}(\delta'^4 z_{0,0} + \delta'^4 z_{0,1} + \delta'^4 z_{1,1} + \delta'^4 z_{1,0}) \\ & - \text{terms depending on differences of sixth order} \\ = {} & {\cdot}884{,}27968 + {\cdot}000{,}80754 + {\cdot}000{,}02594 \\ & \qquad + {\cdot}000{,}00092 + {\cdot}000{,}00010 + {\cdot}000{,}00001 \\ = {} & {\cdot}885{,}1142. \end{aligned}$$

The value obtained by direct calculation is ·885,1140, and the concordance may be considered very satisfactory*.

Now consider how far any other method would be feasible. The table as it stands will occupy about 350 pages of royal octavo or quarto print. Our *first* differences have either five or six significant figures; it appears quite impossible without increasing the table to unprintable dimensions to reduce

* It must be remembered that we have taken the position of nearly maximum error and interpolated for a 0·2 interval instead of the 0·1 interval of the actual table.

Sample of 36 Entries from a Table of Double Entry.

x	$y = 4{\cdot}6$	$4{\cdot}8$	$5{\cdot}0$	$5{\cdot}2$	$5{\cdot}4$	$5{\cdot}6$
	$z_{-2,-2}$	$z_{-2,-1}$	$z_{-2,0}$	$z_{-2,1}$	$z_{-2,2}$	$z_{-2,3}$
3·2	·8131724	·8034645	·7935052	·7833019	·772863	·7621974
δ^2	– 81195	– 82816	– 84209	– 85356	– 86253	– 86882
δ'^2	– 2585	– 2514	– 2440	– 2355	– 2269	– 2173
δ^4	—	—	3513	—	—	—
δ'^4	—	—	11	1	—	—
$\delta^2\delta'^2$	—	—	—	—	—	—
	$z_{-1,-2}$	$z_{1,-1}$	$z_{-1,0}$	$z_{-1,1}$	$z_{-1,2}$	$z_{-1,3}$
3·4	·8530509	·8451570	·8370194	·8286415	·8200272	·8111811
δ^2	– 73028	– 75190	– 77209	– 79075	– 80769	– 82287
δ'^2	– 2465	– 2437	– 2403	– 2364	– 2318	– 2268
δ^4	—	1563	1916	2294	—	—
δ'^4	—	6	4	8	—	—
$\delta^2\delta'^2$	—	143	153	172	—	—
	$z_{0,-2}$	$z_{0,-1}$	$z_{0,0}$	$z_{0,1}$	$z_{0,2}$	$z_{0,3}$
3·6	·8856266	·8793305	·8728127	·8660736	·8591144	·8519361
δ^2	– 63635	– 66001	– 68293	– 70500	– 72617	– 74625
δ'^2	– 2217	– 2217	– 2213	– 2201	– 2191	– 2171
δ^4	212	452	706	972	1272	1576
δ'^4	—	4	8	– 2	—	—
$\delta^2\delta'^2$	—	74	85	90	109	—
	$z_{1,-2}$	$z_{1,-1}$	$z_{1,0}$	$z_{1,1}$	$z_{1,2}$	$z_{1,3}$
3·8	·9118388	·9069039	·9017767	·8964557	·8909399	·8852286
δ^2	– 54030	– 56360	– 58671	– 60953	– 63193	– 65387
δ'^2	– 1909	– 1923	– 1938	– 1948	– 1955	– 1960
δ^4	– 458	– 317	– 151	39	234	454
δ'^4	—	– 1	5	3	—	—
$\delta^2\delta'^2$	—	19	29	42	46	—
	$z_{2,-2}$	$z_{2,-1}$	$z_{2,0}$	$z_{2,1}$	$z_{2,2}$	$z_{2,3}$
4·0	·9326480	·9288413	·9248736	·9207425	·9164461	·9119824
δ^2	– 44883	– 47036	– 49200	– 51367	– 53535	– 55695
δ'^2	– 1585	– 1610	– 1634	– 1653	– 1673	– 1689
δ^4	—	– 782	– 688	– 585	—	—
δ'^4	—	+ 1	– 5	– 1	—	—
$\delta^2\delta'^2$	—	– 11	– 3	– 1	—	—
	$z_{3,-2}$	$z_{3,-1}$	$z_{3,0}$	$z_{3,1}$	$z_{3,2}$	$z_{3,3}$
4·2	·9489689	·9460751	·9430505	·9398926	·9365988	·9331667
δ^2	– 36602	– 38494	– 40417	– 42366	– 44338	– 46328
δ'^2	– 1281	– 1308	– 1333	– 1359	– 1383	– 1405
δ^4	—	—	—	—	—	—
δ'^4	—	—	– 1	+ 2	—	—
$\delta^2\delta'^2$	—	—	—	—	—	—

the intervals so that linear interpolation alone would suffice. If we worked by forward differences we should require to tabulate 14 instead of 5 differences for each tabled value and should then only be correct to the fourth instead of the fifth order of differences. Thus there can be no doubt that the tabulation of central differences in tables of double entry will save at least two-thirds of the difference entries compared with a forward difference tabulation. If it is only necessary to proceed to third order difference accuracy we shall merely require to tabulate two instead of nine differences. The introduction of sixth order differences only contributes five to be subtracted in the *tenth* decimal places and so does not modify the result.

In the above illustration the reader will recognise that a number of the fourth order differences are irregular in the last figure. This must always be the case, if the table has been reduced to, say, seven figures, before the high differences are formed. The raising of the last figure or the dropping of the eighth figure produces such a result. But except in the fact that it is displeasing to the eye it is of no practical importance, owing to the coefficients of the order of $\frac{1}{170}$ to $\frac{1}{250}$ by which these fourth differences are multiplied. In fact for our seven figure result any fourth difference under 20 may virtually be left out of consideration. Even fourth differences of the order 50 will when summed for the angles of the interpolation square only contribute about one unit in the seventh figure.

The object of the present remarks will have been served, if they suggest that where we need to economise space as in tables of double entry, the right method is to use fairly long intervals in the arguments, and tabulate under each function value the five fundamental central differences.

Relation of Mid-panel Central Difference Formulae to Lagrangian Formulae.

We are now in a position to throw light on the nature of the surfaces which may replace the Lagrangian curves of uni-variate interpolation. It will be remembered that Lagrange really puts an nth order parabola through the $n+1$ points he proposes to interpolate from, but no surfaces as simple as such curves suggest themselves for interpolation through 4, 12 and 24 points in general. Now let us approach the solution from the central difference bi-variate formula. To make our problem more general we will suppose our rectangles to be $2a \times 2b$. Then we take

$$\theta = (a-x)/2a, \quad \phi = (a+x)/2a, \quad \chi = (b-y)/2b, \quad \psi = (b+y)/2b.$$

(i) For first differences only we have:

$$z_{xy} = \frac{1}{4}\left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)z_{0,0} + \frac{1}{4}\left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)z_{0,1} + \frac{1}{4}\left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)z_{1,0}$$
$$+\frac{1}{4}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)z_{1,1},$$
$$=\frac{1}{4}(z_{0,0}+z_{0,1}+z_{1,0}+z_{1,1}) + \frac{1}{4}\frac{x}{a}(z_{0,0}+z_{0,1}-z_{1,0}-z_{1,1})$$
$$+\frac{1}{4}\frac{y}{b}(z_{0,0}-z_{0,1}+z_{1,0}-z_{1,1}) + \frac{1}{4}\frac{xy}{ab}(z_{0,0}-z_{0,1}-z_{1,0}+z_{1,1}) \ldots\text{(vii)}.$$

Hence first differences do not give a plane to correspond to the line of linear interpolation; we use a surface of the form

$$y = c_0 + c_1x + c_2y + c_4xy,$$

or a hyperboloid and not a plane. To reduce our general surface of the second degree to a surface fixed by four points only, terms in x^2 and y^2 are discarded.

(ii) For second differences only we have:

$$z_{xy} = \frac{1}{4}\left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left\{1+\frac{1}{8}\left(1-\frac{x^2}{a^2}\right)+\frac{1}{8}\left(1-\frac{y^2}{b^2}\right)\right\}z_{0,0}$$
$$+\frac{1}{4}\left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left\{1+\frac{1}{8}\left(1-\frac{x^2}{a^2}\right)+\frac{1}{8}\left(1-\frac{y^2}{b^2}\right)\right\}z_{0,1}$$
$$+\frac{1}{4}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left\{1+\frac{1}{8}\left(1-\frac{x^2}{a^2}\right)+\frac{1}{8}\left(1-\frac{y^2}{b^2}\right)\right\}z_{1,1}$$
$$+\frac{1}{4}\left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left\{1+\frac{1}{8}\left(1-\frac{x^2}{a^2}\right)+\frac{1}{8}\left(1-\frac{y^2}{b^2}\right)\right\}z_{1,0}$$
$$-\frac{1}{32}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1+\frac{y}{b}\right)z_{1,0} - \frac{1}{32}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1-\frac{y}{b}\right)z_{-1,1}$$
$$-\frac{1}{32}\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\left(1+\frac{x}{a}\right)z_{0,2}$$
$$-\frac{1}{32}\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\left(1-\frac{x}{a}\right)z_{1,2} - \frac{1}{32}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1-\frac{y}{b}\right)z_{2,1}$$
$$-\frac{1}{32}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1+\frac{y}{b}\right)z_{2,0}$$
$$-\frac{1}{32}\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\left(1-\frac{x}{a}\right)z_{1,-1} - \frac{1}{32}\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\left(1+\frac{x}{a}\right)z_{0,1}$$
$$\ldots\ldots\text{(viii)}.$$

This is a quartic equation omitting the terms in x^4, x^2y^2 and y^4. In other words we fit the surface of the form

$$z = c_0 + c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2 + c_6x^3 + c_7x^2y + c_8xy^2 + c_9y^3 + c_{11}x^3y + c_{13}xy^3.$$

(iii) For fourth differences we have the lengthy equation given below. It is clear that in this case we are fitting a sextic with the terms in x^6, x^4y^2, x^2y^4 and y^6 omitted or a surface of the form:

$$\begin{aligned} z = c_0 &+ c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2 + c_6x^3 + c_7x^2y + c_8xy^2 + c_9y^3 \\ &+ c_{10}x^4 + c_{11}x^3y + c_{12}x^2y^2 + c_{13}xy^3 + c_{14}y^4 \\ &+ c_{15}x^5 + c_{16}x^4y + c_{17}x^3y^2 + c_{18}x^2y^3 + c_{19}xy^4 + c_{20}y^5 \\ &+ c_{22}x^5y + c_{24}x^3y^3 + c_{26}xy^5. \end{aligned}$$

It is obvious therefore that in all these cases we make up the requisite number of constants by adding the product of xy and the requisite number of terms in even powers of x and y, i.e.

$$xyc_3, \quad xy(c_{11}x^2 + c_{13}y^2), \quad xy(c_{22}x^4 + c_{24}x^2y^2 + c_{26}y^4), \text{ etc.}$$

It is not obvious why such terms are to be selected. But as the central difference formula gives the form of the Lagrangian parabola in uni-variate interpolation, so by analogy we may expect it to provide the nature of the surface in bi-variate interpolation when we extend the Lagrangian method to this case.

Lagrangian Bi-variate Sextic.

$$\begin{aligned} z_{xy} = z_{0,0}&\left[\frac{1}{256}\left(1+\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left\{\frac{1}{3}\left(9-\frac{x^2}{a^2}\right)\left(25-\frac{x^2}{a^2}\right)+\frac{1}{3}\left(9-\frac{y^2}{b^2}\right)\left(25-\frac{y^2}{b^2}\right)\right.\right. \\ &\qquad\left.\left. -64+\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right\}\right] \\ + z_{0,1}&\left[\frac{1}{256}\left(1+\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left\{\frac{1}{3}\left(9-\frac{x^2}{a^2}\right)\left(25-\frac{x^2}{a^2}\right)+\frac{1}{3}\left(9-\frac{y^2}{b^2}\right)\left(25-\frac{y^2}{b^2}\right)\right.\right. \\ &\qquad\left.\left. -64+\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right\}\right] \\ + z_{1,1}&\left[\frac{1}{256}\left(1-\frac{x}{a}\right)\left(1-\frac{y}{b}\right)\left\{\frac{1}{3}\left(9-\frac{x^2}{a^2}\right)\left(25-\frac{x^2}{a^2}\right)+\frac{1}{3}\left(9-\frac{y^2}{b^2}\right)\left(25-\frac{y^2}{b^2}\right)\right.\right. \\ &\qquad\left.\left. -64+\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right\}\right] \\ + z_{1,0}&\left[\frac{1}{256}\left(1-\frac{x}{a}\right)\left(1+\frac{y}{b}\right)\left\{\frac{1}{3}\left(9-\frac{x^2}{a^2}\right)\left(25-\frac{x^2}{a^2}\right)+\frac{1}{3}\left(9-\frac{y^2}{b^2}\right)\left(25-\frac{y^2}{b^2}\right)\right.\right. \\ &\qquad\left.\left. -64+\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)\right\}\right] \end{aligned}$$

$$
\begin{aligned}
&+ z_{-1,0} \left[\frac{3}{16}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1+\frac{y}{b}\right)\left\{-\frac{1}{48}\left(9-\frac{y^2}{b^2}\right)-\frac{1}{96}\left(25-\frac{x^2}{a^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{-1,1} \left[\frac{3}{16}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1-\frac{y}{b}\right)\left\{-\frac{1}{48}\left(9-\frac{y^2}{b^2}\right)-\frac{1}{96}\left(25-\frac{x^2}{a^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{0,2} \left[\frac{3}{16}\left(1+\frac{x}{a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\left\{-\frac{1}{48}\left(9-\frac{x^2}{a^2}\right)-\frac{1}{96}\left(25-\frac{y^2}{b^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{1,2} \left[\frac{3}{16}\left(1-\frac{x}{a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\left\{-\frac{1}{48}\left(9-\frac{x^2}{a^2}\right)-\frac{1}{96}\left(25-\frac{y^2}{b^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{2,1} \left[\frac{3}{16}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1-\frac{y}{b}\right)\left\{-\frac{1}{48}\left(9-\frac{y^2}{b^2}\right)-\frac{1}{96}\left(25-\frac{x^2}{a^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{2,0} \left[\frac{3}{16}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1+\frac{y}{b}\right)\left\{-\frac{1}{48}\left(9-\frac{y^2}{b^2}\right)-\frac{1}{96}\left(25-\frac{x^2}{a^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{1,-1} \left[\frac{3}{16}\left(1-\frac{x}{a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\left\{-\frac{1}{48}\left(9-\frac{x^2}{a^2}\right)-\frac{1}{96}\left(25-\frac{y^2}{b^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{0,-1} \left[\frac{3}{16}\left(1+\frac{x}{a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\left\{-\frac{1}{48}\left(9-\frac{x^2}{a^2}\right)-\frac{1}{96}\left(25-\frac{y^2}{b^2}\right)+\frac{1}{6}\right\}\right]\\
&+ z_{-2,0} \left[\frac{3}{512}\left(1-\frac{x^2}{9a^2}\right)\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{5a}\right)\left(1+\frac{y}{b}\right)\right]\\
&+ z_{-2,1} \left[\frac{3}{512}\left(1-\frac{x^2}{9a^2}\right)\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{5a}\right)\left(1-\frac{y}{b}\right)\right]\\
&+ z_{0,3} \left[\frac{3}{512}\left(1-\frac{y^2}{9b^2}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{5b}\right)\left(1+\frac{x}{a}\right)\right]\\
&+ z_{1,3} \left[\frac{3}{512}\left(1-\frac{y^2}{9b^2}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{5b}\right)\left(1-\frac{x}{a}\right)\right]\\
&+ z_{3,1} \left[\frac{3}{512}\left(1-\frac{x^2}{9a^2}\right)\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{5a}\right)\left(1-\frac{y}{b}\right)\right]\\
&+ z_{3,0} \left[\frac{3}{512}\left(1-\frac{x^2}{9a^2}\right)\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{5a}\right)\left(1+\frac{y}{b}\right)\right]\\
&+ z_{1,-2} \left[\frac{3}{512}\left(1-\frac{y^2}{9b^2}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{5b}\right)\left(1-\frac{x}{a}\right)\right]\\
&+ z_{0,-2} \left[\frac{3}{512}\left(1-\frac{y^2}{9b^2}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{5b}\right)\left(1+\frac{x}{a}\right)\right]\\
&+ z_{-1,-1} \left[\frac{1}{256}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\right]\\
&+ z_{-1,2} \left[\frac{1}{256}\left(1-\frac{x^2}{a^2}\right)\left(1+\frac{x}{3a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\right]\\
&+ z_{2,2} \left[\frac{1}{256}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1-\frac{y}{3b}\right)\right]\\
&+ z_{2,-1} \left[\frac{1}{256}\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{x}{3a}\right)\left(1-\frac{y^2}{b^2}\right)\left(1+\frac{y}{3b}\right)\right] \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix).}
\end{aligned}
$$

In the general case when we need an interpolation surface passing through a number of points whose projections do not form an equi-rectangular network but in which the ordinate z_r has plan coordinates a_r, b_r we can easily express the coefficients of the z's as determinants, which is probably the easiest method of calculating their value numerically. But the expansion of these determinants does not appear to lead to anything of the simple nature of the Lagrange coefficients in the uni-variate case.

Thus in the simple case of interpolation from four points (z_1, a_1, b_1), (z_2, a_2, b_2), (z_3, a_3, b_3), (z_4, a_4, b_4), we have

$$z_{xy} = z_1 \frac{\begin{vmatrix} x-a_2, & y-b_2, & xy-a_2b_2 \\ a_3-a_2, & b_3-b_2, & a_3b_3-a_2b_2 \\ a_4-a_2, & b_4-b_2, & a_4b_4-a_2b_2 \end{vmatrix}}{\begin{vmatrix} a_1-a_2, & b_1-b_2, & a_1b_1-a_2b_2 \\ a_3-a_2, & b_3-b_2, & a_3b_3-a_2b_2 \\ a_4-a_2, & b_4-b_2, & a_4b_4-a_2b_2 \end{vmatrix}} + z_2 \frac{\begin{vmatrix} x-a_3, & y-b_3, & xy-a_3b_3 \\ a_1-a_3, & b_1-b_3, & a_1b_1-a_3b_3 \\ a_4-a_3, & b_4-b_3, & a_4b_4-a_3b_3 \end{vmatrix}}{\begin{vmatrix} a_2-a_3, & b_2-b_3, & a_2b_2-a_3b_3 \\ a_1-a_3, & b_1-b_3, & a_1b_1-a_3b_3 \\ a_4-a_3, & b_4-b_3, & a_4b_4-a_3b_3 \end{vmatrix}}$$

$$+ z_3 \frac{\begin{vmatrix} x-a_4, & y-b_4, & xy-a_4b_4 \\ a_1-a_4, & b_1-b_4, & a_1b_1-a_4b_4 \\ a_2-a_4, & b_2-b_4, & a_2b_2-a_4b_4 \end{vmatrix}}{\begin{vmatrix} a_3-a_4, & b_3-b_4, & a_3b_3-a_4b_4 \\ a_1-a_4, & b_1-b_4, & a_1b_1-a_4b_4 \\ a_2-a_4, & b_2-b_4, & a_2b_2-a_4b_4 \end{vmatrix}} + z_4 \frac{\begin{vmatrix} x-a_1, & y-b_1, & xy-a_1b_1 \\ a_2-a_1, & b_2-b_1, & a_2b_2-a_1b_1 \\ a_3-a_1, & b_3-b_1, & a_3b_3-a_1b_1 \end{vmatrix}}{\begin{vmatrix} a_4-a_1, & b_4-b_1, & a_4b_4-a_1b_1 \\ a_2-a_1, & b_2-b_1, & a_2b_2-a_1b_1 \\ a_3-a_1, & b_3-b_1, & a_3b_3-a_1b_1 \end{vmatrix}} \quad \text{.........(x).}$$

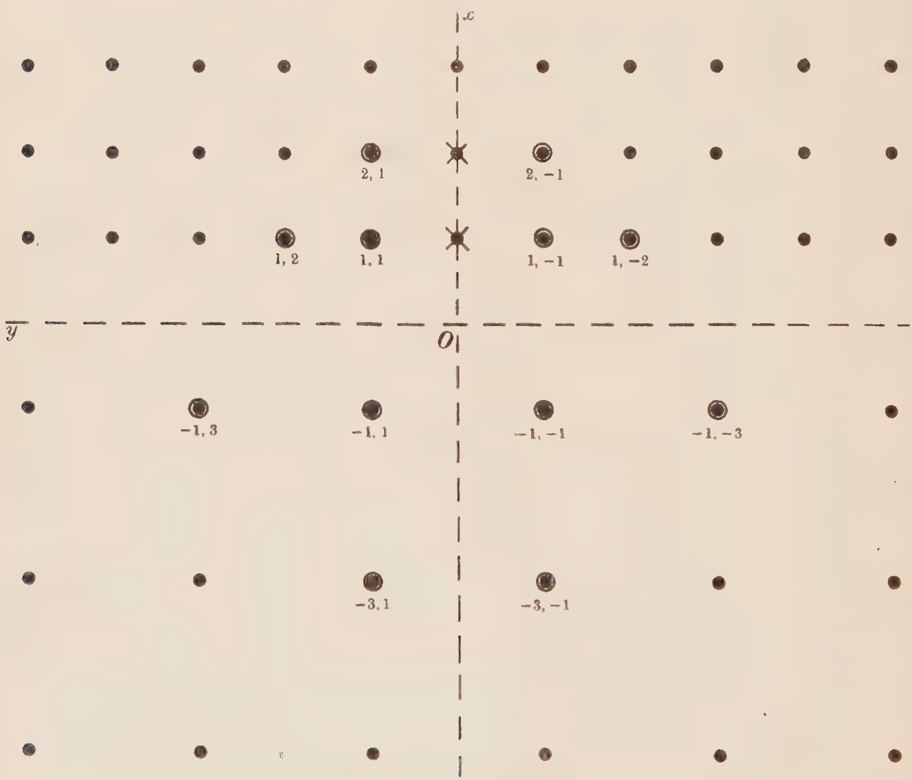
Now it is clear that such formulae, especially the 12 point and 24 point determinantal forms, will scarcely be useful for general interpolation into a table of double entry already calculated. They are too complicated for practical application, just as our formula in (ix) is far inferior to that in (iv). The latter can be used fairly rapidly when once the θ, χ functions are adequately tabled, and when the five central differences are tabled for each function value. On the other hand they may be practically useful when we are filling in a table from its framework. In such a case the values of x, y and of the a's and b's will be the same for a large range of values and z_{xy} will be given in terms of the 12 or 24 adjacent z's and numerical coefficients remaining the same over probably a large area of the table. If we attempted to use the formula in (iv), we should have first to calculate second and fourth differences from the framework, and then discard these and calculate an entirely new system of differences when the table was reduced to its final argument-ranges.

Bi-variate Bridging Formulae.

A similar application of the Lagrange surface type of formulae arises when we need 'bridging formulae.' As we have stated on p. 47 of Tract No. II we understand by a 'bridging formula' one which is used to interpolate the final values of a table when we pass from one section of the preliminary frame with given argument-ranges to a second section in which it has been found needful to increase or decrease one or both argument-ranges of the frame.

For example let it be needful to bridge by interpolation a boundary at which we pass to doubled argument-ranges and we propose to reduce the latter area, or both areas, to decreased argument-ranges.

(vi)



In the accompanying diagram the dots give the plans of the calculated values (i.e. of the elevations). It is required to interpolate in the neighbourhood of O, where Oy is half-way between the two districts. We will suppose

it adequate to use a 12 point formula, and select for our 12 values those of which the plans are enclosed in circles. It will be seen that we have omitted two points—these with crosses through them which are neighbours to O. But if we take these in we shall either have to use 14 points instead of 12, or else omit two others. In the former case we are over-weighting the small argument-range area; in the latter case it is not easy to determine what two points to reject. The system as selected gives a certain semi-symmetry to the distribution in both areas. The result is, of course, of the nature of a compromise, but it appears better to make use of the smaller ranges of the upper area* at least to some extent, as the fact that smaller ranges have been calculated in the frame indicates that the larger ranges are less and less adequate as we pass into this area.

Starting with this assumption of neighbouring points we may take the equation to the required surface (from what we have learnt from our central difference results) to be:

$$z_{xy} = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 + c_{11} x^3 y + c_{13} xy^3.$$

If Ox and Oy be axes, we find for the values of the c's:

$$c_0 = \tfrac{1}{480}(220z_{1,1} + 220z_{1,-1} + 135z_{-1,-1} + 135z_{-1,1} - 48z_{2,1} - 48z_{2,-1} - 40z_{1,2} - 40z_{1,-2} - 12z_{-3,1} - 12z_{-3,-1} - 15z_{-1,3} - 15z_{-1,-3}),$$

$$c_1 = \tfrac{1}{480}(190z_{1,1} + 190z_{1,-1} - 155z_{-1,1} - 155z_{-1,-1} - 16z_{2,1} - 16z_{2,-1} + 6z_{-3,1} + 6z_{-3,-1} - 40z_{1,2} - 40z_{1,-2} + 15z_{-1,3} + 15z_{-1,-3}),$$

$$c_2 = \tfrac{1}{480}(256z_{1,1} - 256z_{1,-1} - 99z_{-1,-1} + 99z_{-1,1} - 12z_{-3,1} + 12z_{-3,-1} - 48z_{2,1} + 48z_{2,-1} - 20z_{1,2} + 20z_{1,-2} - 5z_{-1,3} + 5z_{-1,-3}),$$

$$c_3 = \tfrac{1}{40}(4z_{2,1} + 4z_{2,-1} - 5z_{1,1} - 5z_{1,-1} + z_{-3,1} + z_{-3,-1}),$$

$$c_4 = \tfrac{1}{480}(172z_{1,1} - 172z_{1,-1} + 137z_{-1,-1} - 137z_{-1,1} - 16z_{2,1} + 16z_{2,-1} + 6z_{-3,1} - 6z_{-3,-1} - 20z_{1,2} + 20z_{1,-2} + 5z_{-1,3} - 5z_{-1,-3}),$$

$$c_5 = \tfrac{1}{96}(8z_{1,2} + 8z_{1,-2} - 8z_{1,1} - 8z_{1,-1} - 3z_{-1,-1} - 3z_{-1,1} + 3z_{-1,3} + 3z_{-1,-3}),$$

$$c_6 = \tfrac{1}{240}(8z_{2,1} + 8z_{2,-1} - 15z_{1,1} - 15z_{1,-1} + 10z_{-1,-1} + 10z_{-1,1} - 3z_{-3,1} - 3z_{-3,-1}),$$

$$c_7 = \tfrac{1}{40}(z_{-3,1} - z_{-3,-1} + 4z_{2,1} - 4z_{2,-1} - 8z_{1,1} + 8z_{1,-1} - 3z_{-1,-1} + 3z_{-1,1}),$$

$$c_8 = \tfrac{1}{96}(8z_{1,2} + 8z_{1,-2} - 8z_{1,1} - 8z_{1,-1} + 3z_{-1,-1} + 3z_{-1,1} - 3z_{-1,3} - 3z_{-1,-3}),$$

$$c_9 = \tfrac{1}{96}(4z_{1,2} - 4z_{1,-2} + z_{-1,3} - z_{-1,-3} - 8z_{1,1} + 8z_{1,-1} + 3z_{-1,-1} - 3z_{-1,1}),$$

$$c_{11} = \tfrac{1}{240}(8z_{2,1} - 8z_{2,-1} - 3z_{-3,1} + 3z_{-3,-1} - 6z_{1,1} + 6z_{1,-1} - z_{-1,-1} + z_{-1,1}),$$

$$c_{13} = \tfrac{1}{96}(4z_{1,2} - 4z_{1,-2} - z_{-1,3} + z_{-1,-3} - 8z_{1,1} + 8z_{1,-1} - 3z_{-1,-1} + 3z_{-1,1}) \qquad \text{.........(xi).}$$

* I.e. instead of taking the six upper points arranged like the lower, in which case our 12 points would include nine (instead only of two) unused points within the upper area.

Perhaps the most interesting points for interpolation purposes are those which fill in the values at the centre O and the mid-points of the adjacent square, i.e.

$$z_{0,0} = c_0 = \tfrac{1}{480}(220z_{1,1} + 220z_{1,-1} + 135z_{-1,-1} + 135z_{-1,1} - 48z_{2,1} - 48z_{2,-1}$$
$$- 40z_{1,2} - 40z_{1,-2} - 12z_{-3,1} - 12z_{-3,-1} - 15z_{-1,3} - 15z_{-1,-3}),$$

$$z_{1,0} = c_0 + c_1 + c_3 + c_6 = \tfrac{1}{6}(4z_{1,1} + 4z_{1,-1} - z_{1,2} - z_{1,-2}),$$

$$z_{0,-1} = c_0 - c_2 + c_5 - c_9 = \tfrac{1}{40}(-3z_{1,1} + 33z_{1,-1} + 17z_{-1,-1}$$
$$+ 3z_{-1,1} - 8z_{2,-1} - 2z_{-3,-1}),$$

$$z_{-1,0} = c_0 - c_1 + c_3 - c_6 = \tfrac{1}{16}(9z_{-1,-1} + 9z_{-1,1} - z_{-1,3} - z_{-1,-3}),$$

$$z_{0,1} = c_0 + c_2 + c_5 + c_9 = \tfrac{1}{40}(33z_{1,1} - 3z_{1,-1} + 3z_{-1,-1} + 17z_{-1,1} - 8z_{2,1} - 2z_{-3,1}) \qquad \text{.........(xii).}$$

Now it will be seen that these formulae are of much interest. In the case of $z_{1,0}$ and $z_{-1,0}$ we are thrown back on the uni-variate interpolation formulae, i.e. we interpolate from y-variates only, or from a single row in both of these cases. In the case of $z_{0,-1}$ and $z_{0,1}$ this is, however, not the rule; our interpolation formulae are not merely uni-variate, we use not only the actual column in which we are interpolating, but part of the *adjacent column also.* In other words, the almost universal custom to interpolate into a row or column of a table of double entry by using only values tabled in that row or column may not be really the most effective proceeding. Bi-variate interpolation *may* be best, even when we are interpolating into rows or columns of the actually tabled values. We have not demonstrated that this is so. All we have shown is that in certain cases, apparently depending on the asymmetry of point-distributions, bi-variate interpolation does not break down into uni-variate interpolation, when applied to points actually lying in a row or column of the existing table. It seems to me therefore that when a value has to be interpolated which can be obtained by uni-variate interpolation on a section of the function (treated as a surface) this may give worse results than a bi-variate interpolation from the nearest 4, 12 or 24 points as the case may be.

Illustration. The table on the following page will possibly illustrate the method. We are supposed to be passing to an area of the table where the argument-ranges have been doubled, and we wish to interpolate the values $z_{0,1}$, $z_{0,0}$, $z_{0,-1}$ and $z_{-1,0}$ and also test the accuracy of our work by finding $z_{1,0}$.

The numbers in italics are those made use of for purposes of interpolation.

$(z_{2,3})$	$(z_{2,2})$	$(z_{2,1})$	$(z_{2,0})$	$(z_{2,-1})$	$(z_{2,-2})$	$(z_{2,-3})$
·386,9242	·387,4185	*·387,8742*	·388,2956	*·388,6865*	·389,0501	389,3891·
$(z_{1,3})$	$(z_{1,2})$	$(z_{1,1})$	$(z_{1,0})$	$(z_{1,-1})$	$(z_{1,-2})$	$(z_{1,-3})$
·387,0180	*·387,5085*	*·387,9608*	·388,3790	*·388,7669*	*·389,1277*	·389,4642
		$z_{0,1}$?	$z_{0,0}$?	$z_{0,-1}$?		
$(z_{-1,3})$		$(z_{-1,1})$		$(z_{-1,-1})$		$(z_{-1,-3})$
·387,2102		*·388,1382*	$z_{-1,0}$?	*·388,9317*		*·389,6180*
$(z_{-3,3})$		$(z_{-3,1})$		$(z_{-3,-1})$		$(z_{-3,-3})$
·387,4080		*·388,3208*		*·389,1012*		·389,7762

We find the following results, where we compare together (i) the results of the above formulae, (ii) the results of a four point Lagrange interpolation on the column or row or section, and (iii) the actually calculated values of the function.

	Bi-variate interpolation	Uni-variate interpolation	Actual values
$z_{0,0}$	·388,4639	·388,4639*	·388,4638
$z_{1,0}$	·388,3791	·388,3791	·388,3790
$z_{-1,0}$	·388,5501	·388,5501	·388,5499
$z_{0,1}$	·388,0489†	·388,0488	·388,0488
$z_{0,-1}$	·388,8487†	·388,8487	·388,8487

It will be clear that the results show no advantage from the bi-variate interpolation, when the former differs in form from the uni-variate formula. They only show that the bi-variate interpolation gives quite good results. Let us examine a more critical case.

* To avoid asymmetry we duplicate the formulae for $z_{0,1}$ and $z_{0,-1}$, i.e. take:

$$z_{0,1} = \tfrac{1}{40}(33z_{1,1} - 1{\cdot}5z_{1,-1} - 1{\cdot}5z_{1,3} + 1{\cdot}5z_{-1,-1} + 1{\cdot}5z_{-1,3} + 17z_{-1,1} - 8z_{2,1} - 2z_{-3,1}) \quad \text{(xiii)},$$

$$z_{0,-1} = \tfrac{1}{40}(-1{\cdot}5z_{1,1} - 1{\cdot}5z_{1,-3} + 33z_{1,-1} + 17z_{-1,-1} + 1{\cdot}5z_{-1,1} + 1{\cdot}5z_{-1,-3} - 8z_{2,-1} - 2z_{-3,-1}) \quad \text{(xiv)}.$$

† By diagonal interpolation

$$z_{0,0} = \tfrac{1}{20}(-4z_{2,-2} + 15z_{1,-1} + 10z_{-1,1} - z_{-3,3}),$$

or

$$= \tfrac{1}{20}(-4z_{2,2} + 15z_{1,1} + 10z_{-1,-1} - z_{-3,-3}) \quad \text{(xv)}.$$

Both give the same value to 7 figures.

We take the plan of our values thus:

(vii)

The interpolating points are given by circles and we have the twelve nearest points to the origin O. Uni-variate interpolating formulae will only make use of the points on the axis of x, or those on the axis of y. Or if we made use of both x and y uni-variate interpolations we should use eight points, but they would not be the closest eight points. We need twelve constants for our Lagrangian surface through these twelve points. But, if following up our central difference results we take:

$$z = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y$$
$$+ c_8 xy^2 + c_9 y^3 + c_{11} x^3 y + c_{13} xy^3,$$

we find from the points on the y-axis that

$$c_0 = \tfrac{1}{6}(4z_{0,1} + 4z_{0,-1} - z_{0,2} - z_{0,-2}),$$

or is fully determined by them. If on the other hand we take points on the x-axis

$$c_0 = \tfrac{1}{6}(4z_{1,0} + 4z_{-1,0} - z_{2,0} - z_{-2,0}),$$

and is again fully determined. Accordingly such a surface is impossible unless we have

$$4z_{0,1} + 4z_{0,-1} - z_{0,2} - z_{0,-2} = 4z_{1,0} + 4z_{-1,0} - z_{2,0} - z_{-2,0},$$

a condition which is restrictive and unlikely to hold in any special case.

We have accordingly to choose another pair out of the fourth order terms, and we take $c_{10}x^4$ and $c_{14}y^4$. It is conceivable that some other pair would be more applicable in any special case*. But without special knowledge it seems better to take a homologous pair. We cannot take all the quartic terms for we cannot arrange a 'closest system' of 15 points†.

We start then by passing the surface:

$$z = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2 + c_6 x^3 + c_7 x^2 y + c_8 xy^2 + c_9 y^3 + c_{10} x^4 + c_{14} y^4$$

through our 12 points. We find the constants take the following values:

$$c_0 = \tfrac{1}{4}(2z_{1,0} + 2z_{-1,0} + 2z_{0,1} + 2z_{0,-1} - z_{1,1} - z_{1,-1} - z_{-1,-1} - z_{-1,1}),$$

$$c_1 = \tfrac{1}{12}(8z_{0,1} - 8z_{0,-1} - z_{0,2} + z_{0,-2}), \quad c_2 = \tfrac{1}{12}(8z_{1,0} - 8z_{-1,0} - z_{2,0} + z_{-2,0}),$$

$$c_3 = -\tfrac{1}{48}(2z_{0,2} + 2z_{0,-2} - 2z_{0,1} - 2z_{0,-1} - 15z_{1,1} - 15z_{1,-1} - 15z_{-1,-1} - 15z_{-1,1} + 30z_{1,0} + 30z_{-1,0}),$$

$$c_4 = \tfrac{1}{4}(z_{1,1} - z_{1,-1} + z_{-1,-1} - z_{-1,1}),$$

$$c_5 = -\tfrac{1}{48}(2z_{2,0} + 2z_{-2,0} - 2z_{1,0} - 2z_{-1,0} - 15z_{1,1} - 15z_{1,-1} - 15z_{-1,-1} - 15z_{-1,1} + 30z_{0,1} + 30z_{0,-1}),$$

$$c_6 = -\tfrac{1}{12}(2z_{0,1} - 2z_{0,-1} - z_{0,2} + z_{0,-2}), \quad c_9 = -\tfrac{1}{12}(2z_{1,0} - 2z_{-1,0} - z_{2,0} + z_{-2,0}),$$

$$c_7 = \tfrac{1}{4}(z_{1,1} + z_{1,-1} - z_{-1,-1} - z_{-1,1} - 2z_{1,0} + 2z_{-1,0}),$$

$$c_8 = \tfrac{1}{4}(z_{1,1} - z_{1,-1} - z_{-1,-1} + z_{-1,1} - 2z_{0,1} + 2z_{0,-1}),$$

$$c_{10} = \tfrac{1}{48}(2z_{0,2} + 2z_{0,-2} - 2z_{0,1} - 2z_{0,-1} - 3z_{1,1} - 3z_{1,-1} - 3z_{-1,-1} - 3z_{-1,1} + 6z_{1,0} + 6z_{-1,0}),$$

$$c_{14} = \tfrac{1}{48}(2z_{2,0} + 2z_{-2,0} - 2z_{1,0} - 2z_{-1,0} - 3z_{1,1} - 3z_{1,-1} - 3z_{-1,-1} - 3z_{-1,1} + 6z_{0,1} + 6z_{0,-1}) \quad \text{.........(xvi)}.$$

Clearly $z_{0,0} = c_0$, and it is noteworthy that on the basis of the quartic surface we have selected it involves only the eight closest points and leaves $z_{2,0}$, $z_{-2,0}$, $z_{0,2}$ and $z_{0,-2}$—all requisite for the two uni-variate interpolations—out of account. In order to illustrate numerically I take a table of wide range intervals—i.e. too large for interpolation of the present order—but adequate to show the relative goodness of results. The uni-variate column-interpolation will be:

$$z_{0,0} = \tfrac{1}{6}(4z_{1,0} + 4z_{-1,0} - z_{2,0} - z_{-2,0}),$$

* A pair of terms both involving a power of x (or of y), e.g. $c_{10}x^4 + c_{13}x^2y^2$, appear to be as a rule inappropriate as c_0 will then be a function only of the axial values, i.e. for axial interpolations we are thrown back in one or other case on uni-variate interpolation.

† We could arrange one of 16 points, although not a 'closest system' by using $z_{2,2}$, $z_{2,-2}$, $z_{-2,2}$ and $z_{-2,-2}$; but how is the sixteenth constant to be settled?

and the row-interpolation will be

$$z_{0,0} = \tfrac{1}{6}(4z_{0,1} + 4z_{0,-1} - z_{0,2} - z_{0,-2}),$$

while the bi-variate interpolation is:

$$z_{0,0} = \tfrac{1}{4}(2z_{1,0} + 2z_{-1,0} + 2z_{0,1} + 2z_{0,-1} - z_{1,1} - z_{1,-1} - z_{-1,-1} - z_{-1,1}) \ldots \text{(xvii)}.$$

Extract from table of double entry:

		·344,5302		
	·350,8995	·352,7795	·355,5871	
·356,5570	·357,2083	$z_{0,0}$?	·361,3191	·364,2998
	·362,0547	·363,5239	·365,7144	
		·367,2190,		

$z_{0,0} =$ ·358,8579 from direct calculation,
$z_{0,0} =$ ·358,9107 from column-interpolation, $\Delta = +528$,
$z_{0,0} =$ ·358,8755 from row-interpolation, $\Delta = +176$,
$z_{0,0} =$ ·358,8515 from bi-variate interpolation, $\Delta = -64$.

Thus the bi-variate interpolation is seen to be about three times as good as the row and about eight times as good as the column-interpolation. The uni-variate interpolations are both in excess and differ so considerably from each other that the computer would be somewhat at a loss to choose between them and might average their value with the result that he would be worse than his better value. On the other hand he might select the row-interpolation because the differences are smaller. He would still have a result considerably worse than that from the bi-variate system.

General Mid-point Central Difference Formulae.

Thus far we have been dealing with mid-panel or even number of points interpolation central difference formulae. We will now turn to mid-point or odd number of points central difference formulae. If we put $\phi = 1 - \theta$, and write down the general expression for the uni-variate central difference formula for the mid-panel z_0 to z_1, then write down the same expression for the mid-panel z_0 to z_{-1} changing the sign of θ, we obtain by taking the mean of the two results:

$$\begin{aligned} z_\theta = \;& z_0 + \tfrac{1}{2}\theta(z_1 - z_{-1}) && + \tfrac{1}{2}\theta^2\delta^2 z_0 \\ & - \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_1 - \delta^2 z_{-1}) && - \tfrac{1}{24}\theta^2(1-\theta^2)\delta^4 z_0 \\ & + \tfrac{1}{240}\theta(1-\theta^2)(4-\theta^2)(\delta^4 z_1 - \delta^4 z_{-1}) && + \tfrac{1}{720}\theta^2(1-\theta^2)(4-\theta^2)\delta^6 z_0 \\ & - \frac{\theta(1-\theta^2)(4-\theta^2)(9-\theta^2)}{10080}(\delta^6 z_1 - \delta^6 z_{-1}) && - \tfrac{1}{40320}\theta^2(1-\theta^2)(4-\theta^2)(9-\theta^2)\delta^8 z_0 \\ & + \text{etc.} && + \text{etc.} \end{aligned} \qquad \ldots\ldots\ldots\text{(xviii)}.$$

The terms actually written out give an expression true to eighth differences, or from the Lagrangian standpoint an eighth order parabola passing through the nine points z_0, $z_{\pm 1}$, $z_{\pm 2}$, $z_{\pm 3}$, $z_{\pm 4}$. If we include only the first two lines we have a parabola of the fourth order passing through the five points z_0, $z_{\pm 1}$, $z_{\pm 2}$. We have thus a central difference formula corresponding to an odd number of points, or an even order Lagrangian. The usual central difference formula corresponds to an odd order Lagrangian and the use always of an even number of points. It is true that this formula has a disadvantage over the usual central difference formula, in that it compels us to use *three* entries and their differences, but it is usually more exact in the immediate neighbourhood of a point to the mid-panel central difference formula. We can speak of it as a *mid-point central difference formula.*

It is of course feasible to get rid of one of the three sets of differences, but we must retain the three fundamental table entries. Naturally to keep the formula symmetrical we should get rid of the z_0 differences. For example the mid-point central difference formula true to sixth differences, i.e. corresponding to the Lagrangian through seven points, is

$$\begin{aligned} z_\theta = z_0 (1-\theta^2)(1-\tfrac{1}{9}\theta^2)(1-\tfrac{1}{10}\theta^2) \\ + z_1\ \tfrac{1}{6}\theta(1+\theta)(2+\theta)\{1+\tfrac{1}{30}(1-\theta)(3+\theta)(5-\theta)\} \\ - z_{-1}\tfrac{1}{6}\theta(1-\theta)(2-\theta)\{1+\tfrac{1}{30}(1+\theta)(3-\theta)(5+\theta)\} \\ - \tfrac{1}{360}\theta(1-\theta^2)\{(2+\theta)(3+\theta)(5-\theta)\,\delta^2 z_1 - (2-\theta)(3-\theta)(5+\theta)\,\delta^2 z_{-1}\} \\ + \tfrac{1}{720}\theta(1-\theta^2)(4-\theta^2)\{(3+\theta)\,\delta^4 z_1 - (3-\theta)\,\delta^4 z_{-1}\} \quad \ldots\ldots\ldots\ldots\ldots\text{(xix)}. \end{aligned}$$

We now see the advantages of such a formula. Using fourth differences only of two entries, but three fundamental entries, we are correct up to and including sixth order differences. The usual central difference formula (mid-panel) using fourth differences of two entries, but two fundamental entries only, is only correct up to and including fifth order differences. The disadvantage of such a mid-point central difference formula is the increased complexity of the θ-coefficients. This would be of small importance if they were once tabulated, but we should need to be assured of the advantages of the formula being considerable before undertaking the labour requisite to compute them.

Up to and including fourth order differences we can throw the formula into a form involving only second order differences, namely:

$$\begin{aligned} z_\theta = z_0(1-\theta^2)(1-\tfrac{1}{6}\theta^2) + \tfrac{1}{12}\theta(1+\theta)(2+\theta)(3-\theta)\,z_1 \\ - \tfrac{1}{12}\theta(1-\theta)(2-\theta)(3+\theta)\,z_{-1} \\ - \tfrac{1}{24}\theta(1-\theta^2)\{(2+\theta)\,\delta^2 z_1 - (2-\theta)\,\delta^2 z_{-1}\} \quad \ldots\ldots\ldots\ldots\ldots\text{(xx)}. \end{aligned}$$

Hence by substituting for the remaining differences we reach, as we might anticipate, the 5-point Lagrangian:

$$z_\theta = \tfrac{1}{4}(1-\theta^2)(4-\theta^2)z_0 + \tfrac{1}{6}\theta(1+\theta)(4-\theta^2)z_1 - \tfrac{1}{6}\theta(1-\theta)(4-\theta^2)z_{-1}$$
$$- \tfrac{1}{24}\theta(1-\theta^2)(2+\theta)z_2 + \tfrac{1}{24}\theta(1-\theta^2)(2-\theta)z_{-2} \dots \quad \dots\text{(xxi)}.$$

There are clearly many forms in which such formulae may be written down, and their value will depend on (i) the number of central differences provided with the table, and (ii) the tabling of the θ-coefficients, or their relative simplicity.

Perhaps the simplest form for deducing a bi-variate mid-point central difference formula true up to and including fourth differences is to start from*:

$$z_\theta = z_0(1-\theta^2) + \tfrac{1}{2}\theta(1+\theta)z_1 - \tfrac{1}{2}\theta(1-\theta)z_{-1}$$
$$- \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_1 - \delta^2 z_{-1}) - \tfrac{1}{24}\theta^2(1-\theta^2)\delta^4 z_0 \dots$$

This leads to

$$z_{\theta,\chi} = z_{0,\chi}(1-\theta^2) + \tfrac{1}{2}\theta(1+\theta)z_{1,\chi} - \tfrac{1}{2}\theta(1-\theta)z_{-1,\chi}$$
$$- \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,\chi} - \delta^2 z_{-1,\chi}) - \tfrac{1}{24}\theta^2(1-\theta^2)\delta^4 z_{0,\chi},$$

where we have to substitute:

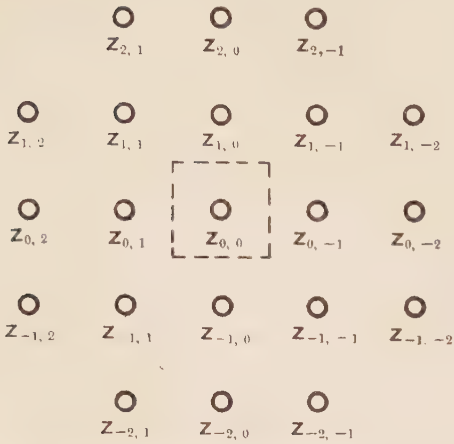
$$z_{0,\chi} = z_{0,0}(1-\chi^2) + \tfrac{1}{2}\chi(1+\chi)z_{0,1} - \tfrac{1}{2}\chi(1-\chi)z_{0,-1}$$
$$- \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}) - \tfrac{1}{24}\chi^2(1-\chi^2)\delta'^4 z_{0,0},$$
$$z_{1,\chi} = z_{1,0}(1-\chi^2) + \tfrac{1}{2}\chi(1+\chi)z_{1,1} - \tfrac{1}{2}\chi(1-\chi)z_{1,-1}$$
$$- \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1}) - \tfrac{1}{24}\chi^2(1-\chi^2)\delta'^4 z_{1,0},$$
$$z_{-1,\chi} = z_{-1,0}(1-\chi^2) + \tfrac{1}{2}\chi(1+\chi)z_{-1,1} - \tfrac{1}{2}\chi(1-\chi)z_{-1,-1}$$
$$- \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{-1,1} - \delta'^2 z_{-1,-1}) - \tfrac{1}{24}\chi^2(1-\chi^2)\delta'^4 z_{-1,0}.$$

After substitution we find:

$$z_{\theta,\chi} = (1-\theta^2)(1-\chi^2)z_{0,0} + \tfrac{1}{2}\chi(1-\theta^2)\{(1+\chi)z_{0,1} - (1-\chi)z_{0,-1}\}$$
$$+ \tfrac{1}{2}\theta(1-\chi^2)\{(1+\theta)z_{1,0} - (1-\theta)z_{-1,0}\}$$
$$+ \tfrac{1}{4}\theta\chi\{(1+\theta)[(1+\chi)z_{1,1} - (1-\chi)z_{1,-1}] - (1-\theta)[(1+\chi)z_{-1,1} - (1-\chi)z_{-1,-1}]\}$$
$$- \tfrac{1}{12}(1-\chi^2)(1-\theta^2)\{\chi(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}) + \theta(\delta^2 z_{1,0} - \delta^2 z_{-1,0})\}$$
$$- \tfrac{1}{24}\theta\chi(1-\chi^2)\{(1+\theta)(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1}) - (1-\theta)(\delta'^2 z_{-1,1} - \delta'^2 z_{-1,-1})\}$$
$$- \tfrac{1}{24}\theta\chi(1-\theta^2)\{(1+\chi)(\delta^2 z_{1,1} - \delta^2 z_{-1,1}) - (1-\chi)(\delta^2 z_{1,-1} - \delta^2 z_{-1,-1})\}$$
$$- \tfrac{1}{48}\theta\chi^2(1-\chi^2)\{(1+\theta)\delta'^4 z_{1,0} - (1-\theta)\delta'^4 z_{-1,0}\}$$
$$- \tfrac{1}{48}\theta^2\chi(1-\theta^2)\{(1+\chi)\delta^4 z_{0,1} - (1-\chi)\delta^4 z_{0,-1}\}$$
$$- \tfrac{1}{24}(1-\theta^2)(1-\chi^2)\{\chi^2\delta'^4 z_{0,0} + \theta^2\delta^4 z_{0,0}\}$$
$$+ \tfrac{1}{144}\chi\theta(1-\chi^2)(1-\theta^2)\{\delta^2\delta'^2 z_{1,1} - \delta^2\delta'^2 z_{1,-1} - \delta^2\delta'^2 z_{-1,1} + \delta^2\delta'^2 z_{-1,-1}\} \quad \dots\text{(xxii)}.$$

* See Tract No. II, p. 20.

(viii)



The sextic (xxii) passes through the 21 points given in the above scheme. The points $z_{\pm 2, \pm 2}$ are not included unless we include higher order terms. Again, the last term written down is of the sixth order of differences and may usually be neglected, the sextic will still pass through the 21 points.

Also terms like $\delta'^4 z_{1,0} - \delta'^4 z_{-1,0}$ and $\delta^4 z_{0,1} - \delta^4 z_{0,-1}$ are of the fifth difference order, and may often be neglected. Further since

$$\delta'^4 z_{1,0} + \delta'^4 z_{-1,0} = 2\delta'^4 z_{0,0} + \delta'^4 \delta^2 z_{0,0},$$
$$\delta^4 z_{0,1} + \delta^4 z_{0,-1} = 2\delta^4 z_{0,0} + \delta^4 \delta'^2 z_{0,0},$$

we may reduce the fourth difference terms to

$$-\tfrac{1}{24}\chi^2(1-\chi^2)\,\delta'^4 z_{0,0} - \tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0} \;\ldots\ldots\ldots\ldots\text{(xxiii)}.$$

When this is done the sextic no longer passes accurately through $z_{2,1}$, $z_{1,2}$, $z_{-1,2}$, $z_{-2,1}$, $z_{-2,-1}$, $z_{-1,-2}$, $z_{1,-2}$, $z_{2,-1}$, but in most cases wherein it is adequate to go to fourth differences, the replacing of the fourth order terms of (xxii) by (xxiii) gives any interpolate within the circumpoint square to the required number of figures. There are many forms in which the result can be given. Thus we may write the apparent first order terms, i.e. the first three lines of (xxii) as:

$$\begin{aligned}
&z_{0,0} + \tfrac{1}{2}\chi(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta(z_{1,0} - z_{-1,0}) + \tfrac{1}{4}\theta\chi(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \\
&\quad + \tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0}) \\
&\quad + \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0} \\
&\quad + \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1}) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxiv)},
\end{aligned}$$

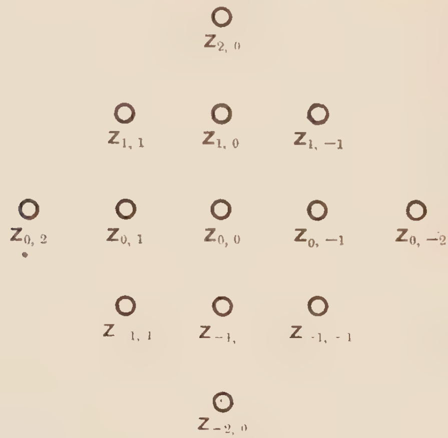
and the apparent second order terms of (xxii), i.e. the fourth, fifth and sixth lines as

$$
\begin{aligned}
&-\tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1}-\delta'^2 z_{0,-1})-\tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,0}-\delta^2 z_{-1,0})\\
&-\tfrac{1}{24}\theta\chi(1-\theta^2)(\delta^2 z_{1,1}-\delta^2 z_{1,-1}-\delta^2 z_{-1,1}+\delta^2 z_{-1,-1})\\
&-\tfrac{1}{24}\theta\chi(1-\chi^2)(\delta'^2 z_{1,1}-\delta'^2 z_{1,-1}-\delta'^2 z_{-1,1}+\delta'^2 z_{-1,-1})\\
&-\tfrac{1}{24}\theta^2\chi(1-\chi^2)\,\delta^2\delta'^2(z_{0,1}-z_{0,-1})\\
&-\tfrac{1}{24}\theta\chi^2(1-\theta^2)\,\delta^2\delta'^2(z_{1,0}-z_{-1,0})\quad\text{......(xxv)}.
\end{aligned}
$$

Here the last two terms of (xxv) are fifth order and may be omitted; the two preceding lines are fourth order, and the first line is third order, or we realise that the apparent second order terms really contribute nothing to the second order formula.

Adding all our results together we obtain a quartic with very simple θ, χ coefficients which passes through the 13 points:

(ix)



and very close to eight more.

Its form is:

$$
\begin{aligned}
z_{\theta,\chi}=z_{0,0}&+\tfrac{1}{2}\chi(z_{0,1}-z_{0,-1})+\tfrac{1}{2}\theta(z_{1,0}-z_{-1,0})+\tfrac{1}{4}\theta\chi(z_{1,1}-z_{1,-1}-z_{-1,1}+z_{-1,-1})\\
&\qquad+\tfrac{1}{2}\chi^2(1-\tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0}+\tfrac{1}{2}\theta^2(1-\tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0}\\
&\qquad+\tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0}+\delta'^2 z_{-1,0}+\delta^2 z_{0,1}+\delta^2 z_{0,-1})\\
&+\tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1}-\delta^2 z_{0,-1})+\tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0}-\delta'^2 z_{-1,0})\\
&-\tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1}-\delta'^2 z_{0,-1})-\tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,0}-\delta^2 z_{-1,0})\\
&-\tfrac{1}{24}\theta\chi(1-\chi^2)(\delta'^2 z_{1,1}-\delta'^2 z_{1,-1}-\delta'^2 z_{-1,1}+\delta'^2 z_{-1,-1})\\
&-\tfrac{1}{24}\theta\chi(1-\theta^2)(\delta^2 z_{1,1}-\delta^2 z_{1,-1}-\delta^2 z_{-1,1}+\delta^2 z_{-1,-1})\\
&-\tfrac{1}{24}\chi^2(1-\chi^2)\,\delta'^4 z_{0,0}-\tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0}\quad\text{......(xxvi)}.
\end{aligned}
$$

The first three lines give the mid-point central difference formula up to second difference terms; the first five lines give it up to third order differences, and, when the remaining three lines are added in, we have the formula up to fourth order difference terms. While involving more differences of the second order than the mid-panel formula it has the advantage of freedom from the cross differences, which only appear as fifth order terms, while they appear as *fourth* order terms in the mid-panel formula.

The reader must be singularly cautious in dealing with multi-variate tables to ascertain that interpolation to fourth order differences is really adequate. The fact that greatly increased argument-ranges are usually requisite in multi-variate tables when compared with uni-variate tables, emphasises the caution needful. In order to be sure that a fourth order difference formula will suffice, it may be needful to examine not only the size of the fifth order difference terms, but those of the sixth order as well, because it may happen that odd order are more influential than even order differences; or conversely if we propose to stop at the fifth order, the seventh may be more important than the sixth order terms. This will be illustrated in the example which follows.

It may therefore be of service to write down the complete value of the mid-point central difference formula up to and including seventh order terms:

$$\left.\begin{aligned}z_{\theta,\chi} = z_{0,0} &+ \tfrac{1}{2}\chi(z_{0,1}-z_{0,-1}) + \tfrac{1}{2}\theta(z_{1,0}-z_{-1,0})\\ &+ \tfrac{1}{4}\theta\chi(z_{1,1}-z_{1,-1}-z_{-1,1}+z_{-1,-1})\end{aligned}\right]\text{1st order}$$

$$\left.\begin{aligned}&+ \tfrac{1}{2}\chi^2(1-\tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1-\tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0}\\ &+ \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0}+\delta'^2 z_{-1,0}+\delta^2 z_{0,1}+\delta^2 z_{0,-1})\end{aligned}\right]\text{2nd order}$$

$$\left.\begin{aligned}&+ \tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1}-\delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0}-\delta'^2 z_{-1,0})\\ &- \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,0}-\delta^2 z_{-1,0}) - \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1}-\delta'^2 z_{0,-1})\end{aligned}\right]\text{3rd order}$$

$$\left.\begin{aligned}&- \tfrac{1}{24}\theta\chi(1-\theta^2)(\delta^2 z_{1,1}-\delta^2 z_{1,-1}-\delta^2 z_{-1,1}+\delta^2 z_{-1,-1})\\ &- \tfrac{1}{24}\theta\chi(1-\chi^2)(\delta'^2 z_{1,1}-\delta'^2 z_{1,-1}-\delta'^2 z_{-1,1}+\delta'^2 z_{-1,-1})\\ &- \tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0} - \tfrac{1}{24}\chi^2(1-\chi^2)\,\delta'^4 z_{0,0}\end{aligned}\right]\text{4th order}$$

$$\left.\begin{aligned}&- \tfrac{1}{48}\theta^2\chi(1-\theta^2)(\delta^4 z_{0,1}-\delta^4 z_{0,-1}) - \tfrac{1}{48}\theta\chi^2(1-\chi^2)(\delta'^4 z_{1,0}-\delta'^4 z_{-1,0})\\ &+ \tfrac{1}{240}\theta(1-\theta^2)(4-\theta^2)(\delta^4 z_{1,0}-\delta^4 z_{-1,0})\\ &+ \tfrac{1}{240}\chi(1-\chi^2)(4-\chi^2)(\delta'^4 z_{0,1}-\delta'^4 z_{0,-1})\\ &- \tfrac{1}{24}\theta\chi^2(1-\theta^2)(\delta^2\delta'^2 z_{1,0}-\delta^2\delta'^2 z_{-1,0})\\ &- \tfrac{1}{24}\theta^2\chi(1-\chi^2)(\delta^2\delta'^2 z_{0,1}-\delta^2\delta'^2 z_{0,-1})\end{aligned}\right]\text{5th order}$$

$$\left.\begin{aligned}
&+\tfrac{1}{720}\theta^2(1-\theta^2)(4-\theta^2)\,\delta^6 z_{0,0}+\tfrac{1}{720}\chi^2(1-\chi^2)(4-\chi^2)\,\delta'^6 z_{0,0}\\
&-\tfrac{1}{48}\theta^2\chi^2(1-\theta^2)\,\delta^4\delta'^2 z_{0,0}-\tfrac{1}{48}\theta^2\chi^2(1-\chi^2)\,\delta^2\delta'^4 z_{0,0}\\
&+\tfrac{1}{144}\theta\chi(1-\theta^2)(1-\chi^2)(\delta^2\delta'^2 z_{1,1}-\delta^2\delta'^2 z_{1,-1}-\delta^2\delta'^2 z_{-1,1}+\delta^2\delta'^2 z_{1,1})\\
&+\tfrac{1}{480}\theta\chi(1-\theta^2)(4-\theta^2)(\delta^4 z_{1,1}-\delta^4 z_{1,-1}-\delta^4 z_{-1,1}+\delta^4 z_{-1,-1})\\
&+\tfrac{1}{480}\theta\chi(1-\chi^2)(4-\chi^2)(\delta'^4 z_{1,1}-\delta'^4 z_{1,-1}-\delta'^4 z_{-1,1}+\delta'^4 z_{-1,-1})
\end{aligned}\right]\ \text{6th order}$$

$$\left.\begin{aligned}
&+\tfrac{1}{1440}\theta^2\chi(1-\theta^2)(4-\theta^2)(\delta^6 z_{0,1}-\delta^6 z_{0,-1})\\
&+\tfrac{1}{1440}\theta\chi^2(1-\chi^2)(4-\chi^2)(\delta'^6 z_{1,0}-\delta'^6 z_{-1,0})\\
&-\tfrac{1}{10080}\theta(1-\theta^2)(4-\theta^2)(9-\theta^2)(\delta^6 z_{1,0}-\delta^6 z_{-1,0})\\
&-\tfrac{1}{10080}\chi(1-\chi^2)(4-\chi^2)(9-\chi^2)(\delta'^6 z_{0,1}-\delta'^6 z_{0,-1})\\
&+\tfrac{1}{288}\theta^2\chi(1-\theta^2)(1-\chi^2)(\delta^4\delta'^2 z_{0,1}-\delta^4\delta'^2 z_{0,-1})\\
&+\tfrac{1}{288}\theta\chi^2(1-\theta^2)(1-\chi^2)(\delta^2\delta'^4 z_{1,0}-\delta^2\delta'^4 z_{-1,0})\\
&+\tfrac{1}{480}\theta\chi^2(1-\theta^2)(4-\theta^2)(\delta^4\delta'^2 z_{1,0}-\delta^4\delta'^2 z_{-1,0})\\
&+\tfrac{1}{480}\theta^2\chi(1-\chi^2)(4-\chi^2)(\delta^2\delta'^4 z_{0,1}-\delta^2\delta'^4 z_{0,-1})
\end{aligned}\right]\ \text{7th order}$$

$$+\ \text{eighth order terms}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxvii)}.$$

We will take as an illustration entries at double argument intervals from a table which at single argument intervals can be adequately interpolated by use of fourth order central differences. The essential part of the table is the following (see p. 12):

	$z_{-1,-1}$	$z_{-1,0}$	$z_{-1,1}$
Entries	*·845,1570*	*·837,0194*	*·828,6415*
Differences δ^2	− 75190	− 77209	− 79075
δ'^2	− 2437	− 2403	− 2364
δ^4	1563	1916	2294
δ'^4	6	4	8
$\delta^2\delta'^2$	143	153	172
	$z_{0,-1}$	$z_{0,0}$	$z_{0,1}$
Entries	*·879,3305*	*·872,8127*	*·866,0736*
Differences δ^2	− 66001	− 68293	− 70500
δ'^2	− 2217	− 2213	− 2201
δ^4	452	706	972
δ'^4	4	8	− 2
$\delta^2\delta'^2$	74	85	90
	$z_{1,-1}$	$z_{1,0}$	$z_{1,1}$
Entries	*·906,9039*	*·901,7767*	*·896,4557*
Differences δ^2	− 56360	− 58671	− 60953
δ'^2	− 1923	− 1938	− 1948
δ^4	− 317	− 151	39
δ'^4	− 1	5	3
$\delta^2\delta'^2$	19	29	42

The rapid fall and change of sign of δ^4 in this region of the table combined with the large argument-ranges makes the interpolation of interest.

Using the formula (xxii) on p. 26 we find, for $\theta = \chi = \frac{1}{2}$,

$$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1152.$$

Using the formula (xxvi) on p. 28 we find

$$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1153,$$

or there is little advantage in the longer over the more approximate formula.

As both the above formulae practically agree while the mid-panel formula gives

$$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1142,$$

it is clear that our neglect in the former cases of fifth order difference terms is the source of our comparative failure.

Now taking formula (xxvii) on p. 29 and noting the successive approximations as we take more and more terms we have:

Up to	first order	terms:	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}6878(00)$,
,,	second	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1858(14)$,
,,	third	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1152(27)$,
,,	fourth*	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1152(99)$,
,,	fifth	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1140(24)$,
,,	sixth	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1140(24)$,
,,	seventh	,,	$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1139(67)$.

The directly computed value is ·885,1140. It will thus be clear that it was needful to include the fifth order terms, but that the sixth and seventh order make no real modification to our seven decimal places. As we might anticipate, the mid-point formula to the same degree of approximation gives slightly better results than the mid-panel formula, for it coincides with the computed value.

We conclude that whenever fifth order differences are negligible mid-point central difference formulae will give satisfactory results, and if a bi-variate table has small enough argument intervals to justify fourth difference interpolation, then we can escape the necessity of tabling $\delta^2\delta'^2$ differences if we use a mid-point and not a mid-panel interpolation.

* This includes the terms in $\frac{1}{24}\theta\chi(1-\theta^2)$ and $\frac{1}{24}\theta\chi(1-\chi^2)$ which are really fifth order terms, i.e. diagonal differences.

On the other hand it is quite easy to write the mid-point interpolation formula (xxvii) of p. 29 up to and including fifth order terms without cross differences. It runs as follows:

$$\begin{aligned}
z_{\theta,\chi} = z_{0,0} &+ \tfrac{1}{2}\chi(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta(z_{1,0} - z_{-1,0}) + \tfrac{1}{4}\theta\chi(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \\
&+ \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0} \\
&+ \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1}) \\
&+ \tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0}) \\
&- \tfrac{1}{12}\theta(1-\theta^2)(1-\chi^2)(\delta^2 z_{1,0} - \delta^2 z_{-1,0}) - \tfrac{1}{12}\chi(1-\theta^2)(1-\chi^2)(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}) \\
&- \tfrac{1}{24}\theta\chi(1-\theta^2)\{(1+\chi)(\delta^2 z_{1,1} - \delta^2 z_{-1,1}) - (1-\chi)(\delta^2 z_{1,-1} - \delta^2 z_{-1,-1})\} \\
&- \tfrac{1}{24}\theta\chi(1-\chi^2)\{(1+\theta)(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1}) - (1-\theta)(\delta'^2 z_{-1,1} - \delta'^2 z_{-1,-1})\} \\
&- \tfrac{1}{48}\theta^2\chi(1-\theta^2)(\delta^4 z_{0,1} - \delta^4 z_{0,-1}) - \tfrac{1}{48}\theta\chi^2(1-\chi^2)(\delta'^4 z_{1,0} - \delta'^4 z_{-1,0}) \\
&+ \tfrac{1}{240}\theta(1-\theta^2)(4-\theta^2)(\delta^4 z_{1,0} - \delta^4 z_{-1,0}) \\
&+ \tfrac{1}{240}\chi(1-\chi^2)(4-\chi^2)(\delta'^4 z_{0,1} - \delta'^4 z_{0,-1}) \\
&- \text{etc.} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{xxviii}).
\end{aligned}$$

On Semi-Lagrangian Bi-variate Interpolation.

We have seen that in uni-variate interpolation a high order parabola is taken, say of the nth order

$$y = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n,$$

and that this is made to pass through the $(n+1)$ *nearest* points, n even corresponds to a mid-point n odd to a mid-panel formula. The method is cumbersome, as it does not show by expression in differences the degree of approximation. It has advantages when the tabular intervals are not equal. We have further seen that when we take bi-variate interpolation—although it is quite possible to write down the extension of the Lagrangian formula to two dimensions—it will not directly give a 'dyadic*' of the lowest possible order passing through the chosen group of nearest points. The dyadics under consideration have for available constants:

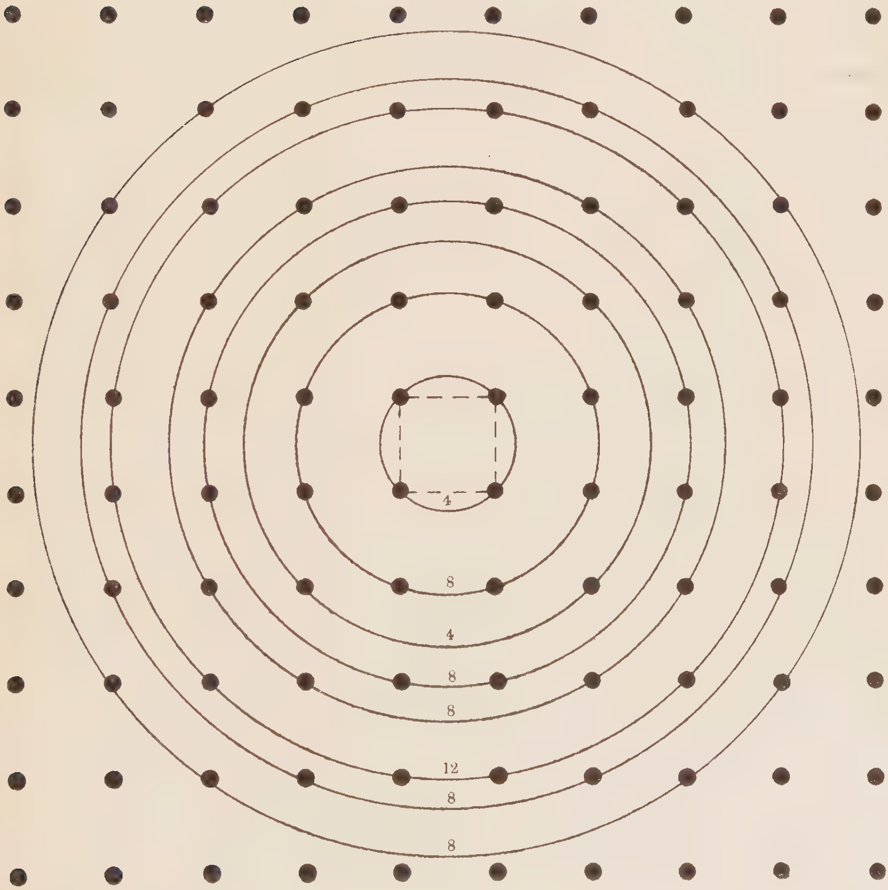
Plane	3	Sextic	28
Quadric	6	Septenic	36
Cubic	10	Octic	45
Quartic	15	Nonic	55
Quintic	21	Decic	66

* I propose to call a surface in two dimensions in which z is given by integer powers of x and y a *dyadic*.

and this represents theoretically the number of points through which they can be made to pass. The number of nearest points are:

Mid-panel system	*Mid-point system*
4	5
12	9
16	13
24	21
32	25
44	29
52	37
60	45

(x)



Now it will be observed that in only two cases, both in the mid-point system, do the number of nearest point systems agree with the number of constants in a dyadic, namely in the case of the quintic and the octic. But it is impossible to take a quintic through the 21 nearest points on a rectangular net-work unless there be a very definite relation among the ordinates (i.e.

(xi)

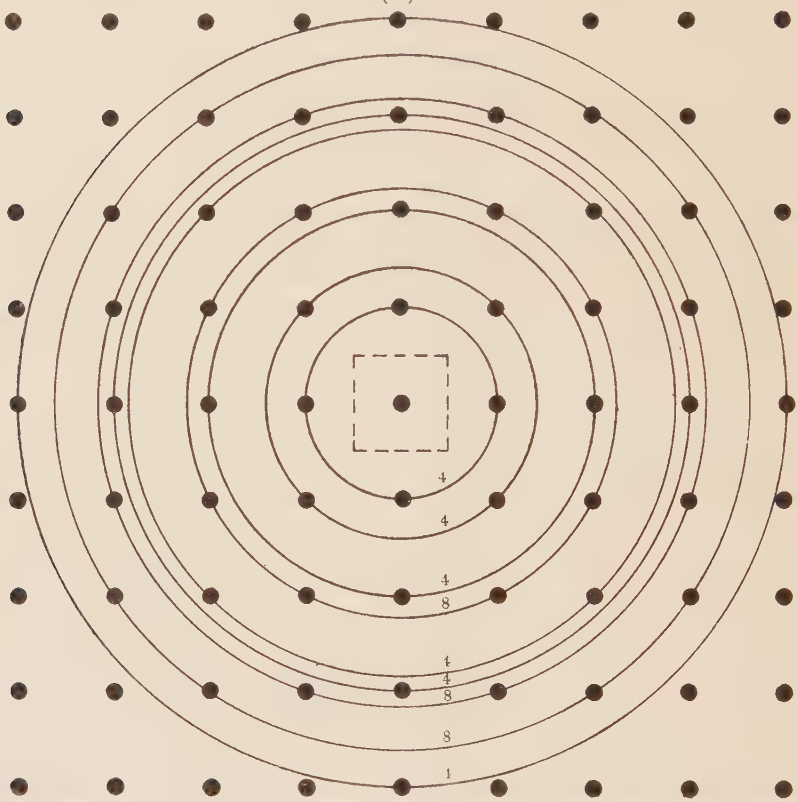


table entries). A like failure occurs in the case of an octic when we try to take it through the 45 nearest points of the mid-point system*. But if it could be taken it would not be of service for practical purposes, as the differences involved are too many for practical interpolation.

If we suppose n points of which the coordinates of the sth are z_s, a_s, b_s, we can at once generalise the Lagrangian and write

$$z_{xy} = \overset{n}{\underset{1}{S}} \frac{(x-a_1)(x-a_2)\dots(x-a_n)(y-b_1)(y-b_2)\dots(y-b_n)\,z_s}{(x-a_s)(y-b_s)(a_s-a_1)(a_s-a_2)\dots(a_s-a_n)(b_s-b_1)(b_s-b_2)\dots(b_s-b_n)}.$$

* When the required relations between the z's hold then the quintic and the octic become incompletely determined by the corresponding 'nearest point' systems.

This is a dyadic of order $2n-2$. For example if we take the 45 nearest points of a mid-point system, this dyadic would be of the 88th order, but theoretically an octic with 45 constants is adequate. In proceeding to the 88th order we have selected only certain terms, neglecting others in an apparently arbitrary manner. Now let us suppose our points form a square in rectangular net-work with n values of x and n values of y. Each term of the n^2 point system must contain $(n-1)\,x$ and $(n-1)\,y$ factors, and our dyadic will be of the $(n-1)^2$ order. For example consider a square of the 7×7 form or 49 points. This will as above require a dyadic of the 36 order. But if an octic will go through 45 points, and a nonic through 55, it appears arbitrary to use certain terms of the 36ic for 49 points.

The non-fitting in the number of constants in the dyadic with the number of nearest points suggests another method of reaching what we may term Semi-Lagrangian bi-variate interpolation formulae. We cannot take a dyadic through more points n than its order m, we therefore take it through fewer points, p, and so use p of its constants; we have $m-p$ of its constants still to use. We select the values of these so as to give the 'best fit' to $n-p$ more points lying on the next ring or on the next two rings. The method used will be that of least squares. A dyadic of the mth order which passes through p nearest points ($< m$) and is the best fit to $n-p$ next nearest points, I accordingly propose to term a Semi-Lagrangian.

It may be suggested that if we take two rings of best fitted points we ought to weight these rings with their distances from the area of interpolation, but we shall only use *two* rings when first the points on them to a certain extent supplement each other, and secondly when the rings are so close that within the area of interpolation some points on the outer ring may be closer to the interpolate than some points on the inner ring. Under the circumstances we believe that weighting would be of little effect or even prejudicial; to be at all correct it would have to vary from point to point as a function of the position of the interpolate.

We will first take two illustrations of Semi-Lagrangians:

(i) A quartic for the mid-point 21-point system. The quartic is taken through the nine-point system and its remaining six constants are determined so that the quartic shall be the closest fit to the additional 12 points of the next two rings of points (see Diagram (xi)). These rings contain respectively four and eight points and their radii are in the ratio of 2 to 2·236, while the adjacent rings have radii of 1·414 and 2·828 respectively; the intervals are therefore

roughly ·6, ·2 and ·6, or it is desirable to keep the 2 and 2·236 rings together. The diagram indicates how the eight points on the outer ring effectually supplement the four points on the inner ring, where the latter are least effectual in determining the form of the surface.

(ii) A quintic for the mid-panel system of 24 points. This was taken through the 12 points of the first two rings (see Diagram (x)), leaving nine disposable constants. These constants were then chosen so as to give the best fit to the 12 additional points on the next two rings. The points on these rings supplement each other, but it would be needful to take another ring with 8 points on it into account—and this did not seem advisable—to get a marked change in radii on either side the best fitted point group. The radii are in the proportions:

$$\underbrace{1{\cdot}414, \quad 3{\cdot}162,}_{\text{Points on surface}} \quad \underbrace{4{\cdot}243, \quad 5{\cdot}099,}_{\text{Best fitted points}} \quad 5{\cdot}831, \quad 7{\cdot}071,$$

showing intervals of:

$$1{\cdot}75, \qquad 1{\cdot}08, \qquad {\cdot}86, \qquad {\cdot}73, \qquad 1{\cdot}24,$$

so that the points on the 4·243, 5·099 and 5·831 circles naturally group together, but nine available constants seemed too few for 20 points to be at all closely fitted.

Mid-point Semi-Lagrangian Quartic.

(i) In the case of the quartic for the mid-point Semi-Lagrangian, the full quartic was written down with its 15 constants and these were determined by least squares subject to the conditions that the quartic should pass through the 9 points. When the final result was reached, however, it was realised that a quicker process would lead to precisely the same result because the terms up to the cubic order were absolutely identical with those of the general mid-point interpolation formula (xxvii) of our p. 29. We can therefore take this as the quadric through the nine points of the first two rings and supplement it by terms in θ, χ of not higher than the fourth order, which are of such a form that their sum vanishes at the nine internal points. The required quartic is accordingly:

$$\begin{aligned} z_{\theta,\chi} = z_{0,0} &+ \tfrac{1}{2}\chi\,(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta\,(z_{1,0} - z_{-1,0}) + \tfrac{1}{4}\theta\chi\,(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \\ &+ \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0} \\ &+ \tfrac{1}{8}\theta^2\chi^2\,(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1}) \\ &+ \tfrac{1}{4}\theta^2\chi\,(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2\,(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0}) \\ &- \tfrac{1}{6}\theta\,(1 - \theta^2)\,(a_1 + a_2\theta + a_3\chi) - \tfrac{1}{6}\chi\,(1 - \chi^2)\,(a_4 + a_5\theta + a_6\chi). \end{aligned}$$

Here a_1, a_2, a_3, a_4, a_5, a_6 are to be determined by the method of least squares from the equations which result when we substitute the twelve points on the two outer rings for $z_{\theta,\chi}$. We find after some reductions:

$$\begin{aligned}
\chi &= 0,\ \theta = 2: & -\delta^2 z_{1,0} + \delta^2 z_{0,0} + a_1 + 2a_2 &= 0,\\
\chi &= 0,\ \theta = -2: & +\delta^2 z_{-1,0} - \delta^2 z_{0,0} + a_1 - 2a_2 &= 0,\\
\chi &= 1,\ \theta = 2: & -\delta^2 z_{1,1} + \delta^2 z_{0,1} + a_1 + 2a_2 + a_3 &= 0,\\
\chi &= 1,\ \theta = -2: & +\delta^2 z_{-1,1} - \delta^2 z_{0,1} + a_1 - 2a_2 + a_3 &= 0,\\
\chi &= -1,\ \theta = 2: & -\delta^2 z_{1,-1} + \delta^2 z_{0,-1} + a_1 + 2a_2 - a_3 &= 0,\\
\chi &= -1,\ \theta = -2: & +\delta^2 z_{-1,-1} - \delta^2 z_{0,-1} + a_1 - 2a_2 - a_3 &= 0.
\end{aligned}$$

Squaring these results and adding the squares of them, we differentiate respectively with regard to a_1, a_2, a_3 and find the resulting equations give at once a_1, a_2 and a_3. We have:

$$\begin{aligned}
a_1 &= \tfrac{1}{6}(\delta^2\delta'^2 z_{1,0} - \delta^2\delta'^2 z_{-1,0}) + \tfrac{1}{2}(\delta^2 z_{1,0} - \delta^2 z_{-1,0}),\\
a_2 &= \tfrac{1}{4}\delta^4 z_{0,0} + \tfrac{1}{12}\delta^4\delta'^2 z_{0,0},\\
a_3 &= \tfrac{1}{4}(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}).
\end{aligned}$$

A precisely similar process provides

$$\begin{aligned}
a_4 &= \tfrac{1}{6}(\delta^2\delta'^2 z_{0,1} - \delta^2\delta'^2 z_{0,-1}) + \tfrac{1}{2}(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}),\\
a_5 &= \tfrac{1}{4}(\delta'^2 z_{1,1} - \delta'^2 z_{-1,1} - \delta'^2 z_{1,-1} + \delta'^2 z_{-1,-1}),\\
a_6 &= \tfrac{1}{4}\delta'^4 z_{0,0} + \tfrac{1}{12}\delta'^4\delta^2 z_{0,0}.
\end{aligned}$$

We can now write down our mid-point quartic Semi-Lagrangian, and the reader is asked to compare it line by line with the formula on p. 29:

$$\begin{aligned}
z_{\theta,\chi} = z_{0,0} &+ \tfrac{1}{2}\chi(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta(z_{1,0} - z_{-1,0})\\
&+ \tfrac{1}{4}\theta\chi(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1})\\
&+ \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\delta^2 z_{0,0}\\
&+ \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1})\\
&+ \tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0})\\
&- \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,0} - \delta^2 z_{-1,0}) - \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1})\\
&- \tfrac{1}{24}\theta\chi(1-\theta^2)(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1})\\
&- \tfrac{1}{24}\chi\theta(1-\chi^2)(\delta'^2 z_{1,1} - \delta'^2 z_{-1,1} - \delta'^2 z_{1,-1} + \delta'^2 z_{-1,-1})\\
&- \tfrac{1}{24}\theta^2(1-\theta^2)\delta^4 z_{0,0} - \tfrac{1}{24}\chi^2(1-\chi^2)\delta'^4 z_{0,0}\\
&- \tfrac{1}{36}\theta(1-\theta^2)(\delta^2\delta'^2 z_{1,0} - \delta^2\delta'^2 z_{-1,0}) - \tfrac{1}{36}\chi(1-\chi^2)(\delta^2\delta'^2 z_{0,1} - \delta^2\delta'^2 z_{0,-1})\\
&- \tfrac{1}{72}\theta^2(1-\theta^2)\delta^4\delta'^2 z_{0,0} - \tfrac{1}{72}\chi^2(1-\chi^2)\delta^2\delta'^4 z_{0,0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xxix)}.
\end{aligned}$$

We started with agreement in the first and second order terms and the first line of the third order terms. Our least square process gives the second line of the third order terms and all the fourth order terms absolutely identical in value with those of the fundamental formula. Of the fifth order terms it preserves only those with cross differences ($\delta^2\delta'^2$), replacing $\frac{1}{24}\theta\chi^2(1-\theta^2)$ and $\frac{1}{24}\theta^2\chi(1-\chi^2)$ by $\frac{1}{36}\theta(1-\theta^2)$ and $\frac{1}{36}\chi(1-\chi^2)$ respectively.

In the sixth order terms it retains only those with cross differences of $z_{0,0}$, replacing the coefficients $\frac{1}{48}\theta^2\chi^2(1-\theta^2)$ and $\frac{1}{48}\theta^2\chi^2(1-\chi^2)$ by $\frac{1}{72}\theta^2(1-\theta^2)$ and $\frac{1}{72}\chi^2(1-\chi^2)$ respectively. Thus we should gain nothing by this mid-point Semi-Lagrangian if we stopped at the fourth order terms. But it does show us that as far as fourth order terms are concerned the general formula not only passes through the nine nearest points, but is the best fit to 12 additional points.

The fifth and sixth order terms are greatly simplified by the Semi-Lagrangian process, but it will always be a question in any special case, if we have to go to fifth and sixth order terms, whether these simplified values are really adequate.

Mid-panel Semi-Lagrangian Quintic.

(ii) In the case of the quintic for the mid-panel Semi-Lagrangian, we start with the first three lines of the formula (iv) on p. 8 and add to it the requisite terms not higher than the fifth order which vanish at the 12 points through which the quintic is to pass. Thus we start with:

$$\begin{aligned}z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & -\tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & -\tfrac{1}{6}\chi\psi\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\} \\ & +\tfrac{1}{24}\theta\phi(1+\theta)(1+\phi)(c_1 + c_2\theta + c_3\chi) \\ & +\tfrac{1}{4}\theta\phi\chi\psi(c_4 + c_5\theta + c_6\chi) \\ & +\tfrac{1}{24}\chi\psi(1+\chi)(1+\psi)(c_7 + c_8\theta + c_9\chi) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xxx)},\end{aligned}$$

where the c's are to be determined so that the quintic fits best to the next nearest 12 points.

Proceeding exactly as before we find three sets, each of four equations, to determine the c's. These are

$$\begin{array}{lll} -\delta^4 z_{0,0} + c_1 - 2c_2 = 0, & -\delta^2\delta'^2 z_{0,0} + c_4 - c_5 - c_6 = 0, & -\delta'^4 z_{0,0} + c_7 - 2c_9 = 0, \\ -\delta^4 z_{1,0} + c_1 + 3c_2 = 0, & -\delta^2\delta'^2 z_{1,0} + c_4 - 2c_5 - c_6 = 0, & -\delta'^4 z_{0,1} + c_7 + 3c_9 = 0, \\ -\delta^4 z_{0,1} + c_1 - 2c_2 + c_3 = 0, & -\delta^2\delta'^2 z_{0,1} + c_4 - c_5 - 2c_6 = 0, & -\delta'^4 z_{1,0} + c_7 + c_8 - 2c_9 = 0, \\ -\delta^4 z_{1,1} + c_1 + 3c_2 + c_3 = 0, & -\delta^2\delta'^2 z_{1,1} + c_4 + 2c_5 + 2c_6 = 0, & -\delta'^4 z_{1,1} + c_7 + c_8 + 3c_9 = 0. \end{array}$$

Hence by least squares we find:

$$c_1 = \tfrac{1}{20}(11\delta^4 z_{0,0} + 9\delta^4 z_{1,0} + \delta^4 z_{0,1} - \delta^4 z_{1,1}),$$
$$c_2 = \tfrac{1}{10}(-\delta^4 z_{0,0} + \delta^4 z_{1,0} - \delta^4 z_{0,1} + \delta^4 z_{1,1}),$$
$$c_3 = \tfrac{1}{2}(-\delta^4 z_{0,0} - \delta^4 z_{1,0} + \delta^4 z_{0,1} + \delta^4 z_{1,1}),$$
$$c_4 = \tfrac{1}{12}(5\delta^2\delta'^2 z_{0,0} + 3\delta^2\delta'^2 z_{1,0} + 3\delta^2\delta'^2 z_{0,1} + \delta^2\delta'^2 z_{1,1}),$$
$$c_5 = \tfrac{1}{6}(-\delta^2\delta'^2 z_{0,0} + \delta^2\delta'^2 z_{1,0} - \delta^2\delta'^2 z_{0,1} + \delta^2\delta'^2 z_{1,1}),$$
$$c_6 = \tfrac{1}{6}(-\delta^2\delta'^2 z_{0,0} - \delta^2\delta'^2 z_{1,0} + \delta^2\delta'^2 z_{0,1} + \delta^2\delta'^2 z_{1,1}),$$
$$c_7 = \tfrac{1}{20}(11\delta'^4 z_{0,0} + 9\delta'^4 z_{0,1} + \delta'^4 z_{1,0} - \delta'^4 z_{1,1}),$$
$$c_8 = \tfrac{1}{2}(-\delta'^4 z_{0,0} - \delta'^4 z_{0,1} + \delta'^4 z_{1,0} + \delta'^4 z_{1,1}),$$
$$c_9 = \tfrac{1}{10}(-\delta'^4 z_{0,0} + \delta'^4 z_{0,1} - \delta'^4 z_{1,0} + \delta'^4 z_{1,1}).$$

Hence, remembering that $\theta + \phi = 1$, $\chi + \psi = 1$, we reach for our Semi-Lagrangian quintic:

$$\begin{aligned}
z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\
& - \tfrac{1}{6}\theta\phi\,\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\
& - \tfrac{1}{6}\chi\psi\,\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\} \\
& + \tfrac{1}{480}\theta\phi(1+\theta)(1+\phi)\,\{(2\phi + 10\psi - 1)\,\delta^4 z_{0,0} + (2\theta + 10\psi - 1)\,\delta^4 z_{1,0} \\
& \qquad + (2\phi + 10\chi - 1)\,\delta^4 z_{0,1} + (2\theta + 10\chi - 1)\,\delta^4 z_{1,1}\} \\
& + \tfrac{1}{48}\theta\phi\chi\psi\,\{(1 + 2\phi + 2\psi)\,\delta^2\delta'^2 z_{0,0} + (1 + 2\phi + 2\chi)\,\delta^2\delta'^2 z_{0,1} \\
& \qquad + (1 + 2\theta + 2\psi)\,\delta^2\delta'^2 z_{1,0} + (1 + 2\theta + 2\chi)\,\delta^2\delta'^2 z_{1,1}\} \\
& + \tfrac{1}{480}\chi\psi(1+\chi)(1+\psi)\,\{(2\psi + 10\phi - 1)\,\delta'^4 z_{0,0} + (2\chi + 10\phi - 1)\,\delta'^4 z_{0,1} \\
& \qquad + (2\psi + 10\theta - 1)\,\delta'^4 z_{1,0} + (2\chi + 10\theta - 1)\,\delta'^4 z_{1,1}\} \quad \ldots\text{(xxxi)}.
\end{aligned}$$

If we take the mid-point of the centre panel, i.e. $\theta = \phi = \chi = \psi = \frac{1}{2}$, the present formula coincides with the general formula (iv) on p. 8, up to but not including terms involving differences of the sixth order. But unlike the mid-point Semi-Lagrangian Quartic, the terms up to the fifth order are not the same in the fundamental and least square formulae. Further, if we exclude sixth order difference terms, our present Semi-Lagrangian is no improvement in simplicity on the fundamental formula of p. 8. It is thus of no greater ease in computation. What then is the difference between the two? Why, the fundamental formula is a *sextic* passing through the 24 points, while the Semi-Lagrangian is a *quintic* passing through 12 of the points and the best fit to the remaining 12 through which the sextic actually passes. Is there any theoretical reason why a dyadic of lower order should give *ceteris paribus* a better interpolation than one of higher order? The question might possibly be best answered by considering the corresponding problem in uni-variate

interpolation: Is there *a priori* any reason, if we are going to use five points for interpolation, why

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$$

should give a better result than

$$y = c_0 + c_1x + c_3x^3 + c_4x^4 + c_5x^5,$$

where the x^2 or any other term is omitted and we introduce an additional power? The true answer may lie in the nature of the function to be interpolated. Or, can we say that that interpolating curve (or surface) will be as a rule *a priori* better in which only the highest differences or differentials are *ab initio* determined? If such a principle could be established then we might have a theoretical ground for preferring a Semi-Lagrangian quartic or quintic to dyadics of higher order which actually pass through a larger number of nearest points.

Comparison in the case of Bi-variate Interpolation of results of Uni-variate Interpolation and 'nearest points' Interpolation.

Another matter is also again brought under consideration by the discussion of Semi-Lagrangians.

Let us put $\chi = 0$, or interpolate on the x-axis. The mid-point formula of p. 29 reduces to

$$z_{\theta,0} = z_{0,0} + \tfrac{1}{2}\theta\,(z_{1,0} - z_{-1,0}) + \tfrac{1}{2}\theta^2\delta^2 z_{0,0} - \tfrac{1}{12}\theta\,(1-\theta^2)\,(\delta^2 z_{1,0} - \delta^2 z_{-1,0})$$
$$- \tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0} + \tfrac{1}{240}\theta\,(1-\theta^2)\,(4-\theta^2)\,(\delta^4 z_{1,0} - \delta^4 z_{-1,0}),$$

in other words it becomes uni-variate and involves only points on the x-axis.

The Semi-Lagrangian quartic, however, gives us

$$z_{\theta,0} = z_{0,0} + \tfrac{1}{2}\theta\,(z_{1,0} - z_{-1,0}) + \tfrac{1}{2}\theta^2\delta^2 z_{0,0} - \tfrac{1}{12}\theta\,(1-\theta^2)\,(\delta^2 z_{1,0} - \delta^2 z_{-1,0})$$
$$- \tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0} - \tfrac{1}{36}\theta\,(1-\theta^2)\,(\delta^2\delta'^2 z_{1,0} - \delta^2\delta'^2 z_{-1,0})$$
$$- \tfrac{1}{72}\theta^2(1-\theta^2)\,\delta^4\delta'^2 z_{0,0},$$

and the operator δ'^2 means that we interpolate not only by aid of points on the axis of x, but of points on the adjacent columns $\chi = 1$ and $\chi = -1$ as well. In other words the comparative value of the general mid-point formula and the Semi-Lagrangian raises again the question of whether it is best to use points on one section, or nearest points. The general mid-point formula on the axis of x involves the use of the points $z_{3,0}, z_{2,0}, z_{1,0}, z_{0,0}, z_{-1,0}, z_{-2,0}, z_{-3,0}$. The Semi-Lagrangian does not use the more distant $z_{3,0}$ and $z_{-3,0}$, but the 15 points, five to each column

$$\begin{array}{ccccc} z_{2,1}, & z_{1,1}, & z_{0,1}, & z_{-1,1}, & z_{-2,1}, \\ z_{2,0}, & z_{1,0}, & z_{0,0}, & z_{-1,0}, & z_{-2,0}, \\ z_{2,-1}, & z_{1,-1}, & z_{0,-1}, & z_{-1,-1}, & z_{-2,-1}. \end{array}$$

In the same manner, when $\chi = 0$, the mid-panel general formula (iv) of p. 8 becomes:

$$z_{\theta,0} = \phi z_{0,0} + \theta z_{1,0} - \tfrac{1}{6}\theta\phi\{(1+\phi)\,\delta^2 z_{0,0} + (1+\theta)\,\delta^2 z_{1,0}\}$$
$$+ \tfrac{1}{120}\theta\phi(1+\theta)(1+\phi)\{(2+\phi)\,\delta^4 z_{0,0} + (2+\theta)\,\delta^4 z_{1,0}\} + \dots,$$

and involves the six points on the section $\chi = 0$, namely

$$z_{3,0},\ z_{2,0},\ z_{1,0},\ z_{0,0},\ z_{-1,0},\ z_{-2,0}.$$

But the semi-Lagrangian quintic is:

$$z_{\theta,0} = \phi z_{0,0} + \theta z_{1,0} - \tfrac{1}{6}\theta\phi\{(1+\phi)\,\delta^2 z_{0,0} + (1+\theta)\,\delta^2 z_{1,0}\}$$
$$+ \tfrac{1}{480}\theta\phi(1+\theta)(1+\phi)\{(9+2\phi)\,\delta^4 z_{0,0} + (9+2\theta)\,\delta^4 z_{1,0}$$
$$+ (\phi-\theta)(\delta^4 z_{0,1} - \delta^4 z_{1,1})\}.$$

It therefore involves not only $z_{3,0}, z_{2,0}, z_{1,0}, z_{0,0}, z_{-1,0}, z_{-2,0}$ but also $z_{3,1}, z_{2,1}, z_{1,1}, z_{0,1}, z_{-1,1}, z_{-2,1}$. It thus involves two columns of six points. It is conceivable that this use of side row points may give certain advantages to the Semi-Lagrangian. As a matter of fact if we were interpolating on the boundary ($\chi = 0$) we should certainly take the mean of two Semi-Lagrangians, i.e. that for mid-panel $z_{0,1}, z_{0,0}, z_{1,0}, z_{1,-1}$ as well as that for mid-panel $z_{0,0}, z_{1,0}, z_{1,1}, z_{0,1}$. Thus we should use also the points

$$z_{3,-1},\quad z_{2,-1},\quad z_{1,-1},\quad z_{0,-1},\quad z_{1,-1},\quad z_{2,-1},$$

or three columns of six points each.

These points would of course occur only in fourth and fifth order differences, and the Semi-Lagrangian Quintic may not give them with the best possible numerical factors, yet these Semi-Lagrangians again raise the question of whether good interpolating formulae for bi-variate functions ought to be deduced directly from uni-variate formulae since such deduction invariably leads to the re-appearance of uni-variate formulae along the axes, and so contradicts the axiom that we ought to use the nearest points in interpolation. The formulae given in this part of our tract for bi-variate interpolation we know by experience to give adequate accuracy. But it cannot be said that the question of whether n points in a uni-variate section will give a better or worse result than the n nearest points has been elucidated, and it is to be hoped that it will receive more complete treatment from some mathematician by profession.

Let us illustrate our Semi-Lagrangians on the example of p. 30. Down to the fourth order terms both are in perfect agreement and give

$$z_{0\cdot5,\,0\cdot5} = \cdot885{,}1152\,(99).$$

But while the fifth order terms in the fundamental mid-point formula reduce this value by 12 (75) to the correct ·885,1140 of direct computation and sixth and seventh order terms do not modify the result, the mid-panel Semi-Lagrangian *raises* this value by 1 (13) for its fifth order terms and leaves it unmodified by its sixth order terms. Thus the Semi-Lagrangian fails precisely in the terms where it differs from and abbreviates the fundamental formula. At the same time it must be recorded that we have taken the most unfavourable case possible, i.e. one in which the interpolate is on the extreme limit of the area to which the formula applies.

We will now compare the result obtained by the Semi-Lagrangian Quintic with the fundamental mid-panel formula (iv) of p. 8.

	Semi-Lagrangian Quintic	*Semi-Lagrangian Quartic*
To 1st order	·884,2796 (75)	·885,6878 (00)
„ 2nd „	·885,1131 (66)	·885,1858 (14)
„ 3rd „	—	·885,1152 (27)
„ 4th „	·885,1141 (87)	·885,1152 (99)
„ 5th „	—	·885,1153 (12)

Thus we see that the Semi-Lagrangian Quintic gives the same value, ·885,1142, as the fundamental mid-panel formula. Further to the second order terms both are more correct than the Semi-Lagrangian Quartic or the fundamental mid-point formula. To the fourth order terms the quintic is more correct appreciably than the quartic. But the most satisfactory result is that of the mid-point fundamental formula which is absolutely correct at fifth order terms. When we remember that our table intervals in this illustration are the doubles of the actual table intervals, which were adopted in order to use effectively fourth order difference formulae, the results of all the formulae cannot be considered as unsatisfactory. In saying this we must remind the reader that we have selected a case which tells most in favour of the mid-panel formulae, for the point is in the centre of the interpolation area, and least in favour of the mid-point formulae, for the point is at the very extreme of the interpolation area. The order of the efficiency under these circumstances is:

(i) Mid-point Fundamental Formula.

(ii) Mid-panel Fundamental Formula.
(iii) Semi-Lagrangian Mid-panel Quintic.

(iv) Semi-Lagrangian Mid-point Quartic.

We cannot assert, merely on the basis of this single illustration, that this will always be so, but it has an *a priori* suggestion of reasonableness about it. The success of the Mid-point Fundamental Formula suggested that it would be worth trying a Semi-Lagrangian Quintic passing through 13 points and using the remaining 8 constants to fit it as closely as possible to the next nearest 12 or perhaps 16 points. This we will take as a final illustration of Semi-Lagrangian interpolation.

Mid-point Semi-Lagrangian Quintic.

We will take this through the 13 nearest points and use our remaining 8 constants to give the best fit to 16 further points arranged on the circle of 8 and two circles of 4 (see Diagram (xi)). Now if the general dyadic of the fifth order in θ and χ be written down with its 21 constants and be made to be zero at the mid-point and the 12 next nearest its form is found to be:

$$\begin{aligned} z_{\theta,\chi} = {} & c_1\theta\chi(1-\theta^2) + c_2\theta\chi(1-\chi^2) + c_3\theta^2\chi(1-\theta^2) + c_4\theta\chi^2(1-\chi^2) \\ & + c_5\theta^2\chi(1-\chi^2) + c_6\theta\chi^2(1-\theta^2) + c_7\theta(1-\theta^2)(4-\theta^2) + c_8\chi(1-\chi^2)(4-\chi^2), \end{aligned}$$

and if this be added to the value of $z_{\theta,\chi}$ given on p. 28, which corresponds to a quartic through the 13 points, we obtain the most general form of the quintic through the 13 points with 8 constants to be chosen so as to give the best fit to the next nearest 16 points. Accordingly we start with the quintic

$$\begin{aligned} z_{\theta,\chi} = {} & z_{0,0} + \tfrac{1}{2}\chi(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta(z_{1,0} - z_{-1,0}) + \tfrac{1}{4}\theta\chi(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \\ & + \tfrac{1}{2}\chi^2(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0} \\ & \qquad + \tfrac{1}{8}\theta^2\chi^2(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1}) \\ & + \tfrac{1}{4}\theta^2\chi(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0}) \\ & \qquad - \tfrac{1}{12}\chi(1-\chi^2)(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}) - \tfrac{1}{12}\theta(1-\theta^2)(\delta^2 z_{1,0} - \delta^2 z_{-1,0}) \\ & - \tfrac{1}{24}\theta\chi(1-\chi^2)(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) \\ & \qquad - \tfrac{1}{24}\theta\chi(1-\theta^2)(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) \\ & \qquad - \tfrac{1}{24}\chi^2(1-\chi^2)\,\delta'^4 z_{0,0} - \tfrac{1}{24}\theta^2(1-\theta^2)\,\delta^4 z_{0,0} \\ & + \theta\chi(1-\theta^2)(c_1 + c_3\theta + c_6\chi) + \theta\chi(1-\chi^2)(c_2 + c_5\theta + c_4\chi) \\ & \qquad + c_7\theta(1-\theta^2)(4-\theta^2) + c_8\chi(1-\chi^2)(4-\chi^2) \quad \ldots\ldots\text{(xxxii)}. \end{aligned}$$

The following are the 16 next nearest points and the resulting equations:

$\theta = 2,\ \chi = +1$
$$6(c_1 + 2c_3 + c_6) = \tfrac{1}{4}(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) + \delta^2 z_{1,0} + \delta^2 z_{0,1} - \delta^2 z_{1,1} - \delta^2 z_{0,0},$$

$\theta = 2,\ \chi = -1$
$$-6(c_1 + 2c_3 - c_6) = -\tfrac{1}{4}(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) + \delta^2 z_{1,0} + \delta^2 z_{0,-1} - \delta^2 z_{1,-1} - \delta^2 z_{0,0},$$

$\theta = -2,\ \chi = +1$
$$-6(c_1 - 2c_3 + c_6) = -\tfrac{1}{4}(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) + \delta^2 z_{-1,0} + \delta^2 z_{0,1} - \delta^2 z_{-1,1} - \delta^2 z_{0,0},$$

$\theta = -2,\ \chi = -1$
$$6(c_1 - 2c_3 - c_6) = \tfrac{1}{4}(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) + \delta^2 z_{-1,0} + \delta^2 z_{0,-1} - \delta^2 z_{-1,-1} - \delta^2 z_{0,0},$$

$\theta = 1,\ \chi = 2$
$$6(c_2 + c_5 + 2c_4) = \tfrac{1}{4}(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) + \delta'^2 z_{0,1} + \delta'^2 z_{1,0} - \delta'^2 z_{1,1} - \delta^2 z_{0,0},$$

$\theta = 1,\ \chi = -2$
$$-6(c_2 + c_5 - 2c_4) = -\tfrac{1}{4}(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) + \delta'^2 z_{1,0} + \delta'^2 z_{0,-1} - \delta'^2 z_{1,-1} - \delta^2 z_{0,0},$$

$\theta = -1,\ \chi = 2$
$$-6(c_2 - c_5 + 2c_4) = -\tfrac{1}{4}(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) + \delta'^2 z_{0,1} + \delta'^2 z_{-1,0} - \delta'^2 z_{-1,1} - \delta^2 z_{0,0},$$

$\theta = -1,\ \chi = -2$
$$6(c_2 - c_5 - 2c_4) = \tfrac{1}{4}(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) + \delta'^2 z_{-1,0} + \delta'^2 z_{0,-1} - \delta'^2{}_{-1,-1} - \delta^2 z_{0,0},$$

$\theta = 2,\ \chi = 2$
$$12\{c_1 + c_2 + 2(c_3 + c_5) + 2(c_4 + c_6)\} = \delta^2\delta'^2(z_{0,0} - z_{1,1}) + (\delta^2 + \delta'^2)\{2(z_{0,1} + z_{1,0} - z_{1,1} - z_{0,0}) + \tfrac{1}{2}(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1})\},$$

$\theta = 2,\ \chi = -2$
$$12\{-c_1 - c_2 - 2(c_3 + c_5) + 2(c_4 + c_6)\} = \delta^2\delta'^2(z_{0,0} - z_{1,-1}) + (\delta^2 + \delta'^2)\{2(z_{1,0} + z_{0,-1} - z_{1,-1} - z_{0,0}) - \tfrac{1}{2}(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1})\},$$

$\theta = -2,\ \chi = 2$
$$12\{-c_1 - c_2 + 2(c_3 + c_5) - 2(c_4 + c_6)\} = \delta^2\delta'^2(z_{0,0} - z_{-1,1}) + (\delta^2 + \delta'^2)\{2(z_{0,1} + z_{-1,0} - z_{-1,1} - z_{0,0}) - \tfrac{1}{2}(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1})\},$$

$\theta = -2,\ \chi = -2$
$$12\{c_1 + c_2 - 2(c_3 + c_5) - 2(c_4 + c_6)\} = \delta^2\delta'^2(z_{0,0} - z_{-1,-1}) + (\delta^2 + \delta'^2)\{2(z_{-1,0} + z_{0,-1} - z_{-1,-1} - z_{0,0}) + \tfrac{1}{2}(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1})\},$$

$\theta = 3,\ \chi = 0$ $\quad c_7 = \tfrac{1}{120}(\delta^4 z_{1,0} - \delta^4 z_{0,0}),$

$\theta = -3,\ \chi = 0$ $\quad c_7 = -\tfrac{1}{120}(\delta^4 z_{-1,0} - \delta^4 z_{0,0}),$

$\theta = 0,\ \chi = 3$ $\quad c_8 = \tfrac{1}{120}(\delta'^4 z_{0,1} - \delta'^4 z_{0,0}),$

$\theta = 0,\ \chi = -3$ $\quad c_8 = -\tfrac{1}{120}(\delta'^4 z_{0,-1} - \delta'^4 z_{0,0}).$

The last four equations give us immediately

$$c_7 = \tfrac{1}{240}\delta^4(z_{1,0} - z_{-1,0}) \quad \text{.....................(xxxiii)},$$
$$c_8 = \tfrac{1}{240}\delta'^4(z_{0,1} - z_{0,-1}) \quad \text{.....................(xxxiv)}.$$

Let us call the right-hand side of the first 12 of the above equations, $A, B, C, D, E, F, G, H, K, L, M, N$ respectively. Then to apply the method of least squares we have:

$$\begin{aligned}u^2 = {} & (6c_1 + 12c_3 + 6c_6 - A)^2 + (-6c_1 - 12c_3 + 6c_6 - B)^2 \\ & + (-6c_1 + 12c_3 - 6c_6 - C)^2 + (6c_1 - 12c_3 - 6c_6 - D)^2 \\ & + (6c_2 + 6c_5 + 12c_4 - E)^2 + (-6c_2 - 6c_5 + 12c_4 - F)^2 \\ & + (-6c_2 + 6c_5 - 12c_4 - G)^2 + (6c_2 - 6c_5 - 12c_4 - H)^2 \\ & + (12c_1 + 12c_2 + 24c_3 + 24c_5 + 24c_4 + 24c_6 - K)^2 \\ & + (-12c_1 - 12c_2 - 24c_3 - 24c_5 + 24c_4 + 24c_6 - L)^2 \\ & + (-12c_1 - 12c_2 + 24c_3 + 24c_5 - 24c_4 - 24c_6 - M)^2 \\ & + (12c_1 + 12c_2 - 24c_3 - 24c_5 - 24c_4 - 24c_6 - N)^2;\end{aligned}$$

and must differentiate with regard to the c's. First, $\dfrac{du^2}{dc_1} = 0$ and $\dfrac{du^2}{dc_2} = 0$ lead at once to

$$5c_1 + 4c_2 = \tfrac{1}{24}(A - B - C + D) + \tfrac{1}{12}(K - L - M + N),$$
$$4c_1 + 5c_2 = \tfrac{1}{24}(E - F - G + H) + \tfrac{1}{12}(K - L - M + N).$$

But $A - B - C + D = 0$ and $E - F - G + H = 0$, while

$$K - L - M + N = -\delta^2\delta'^2(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}).$$

Thus:

$$c_1 = -\tfrac{1}{108}\delta^2\delta'^2(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \text{ ....(xxxv)},$$
$$c_2 = -\tfrac{1}{108}\delta^2\delta'^2(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \text{ ....(xxxvi)}.$$

Again $\dfrac{du^2}{dc_3} = 0$ and $\dfrac{du^2}{dc_5} = 0$ give us:

$$5c_3 + \ \ 4c_5 = \tfrac{1}{48}(A - B + C - D) + \tfrac{1}{24}(K - L + M - N),$$
$$16c_3 + 17c_5 = \tfrac{1}{24}(E - F + G - H) + \tfrac{1}{6}(K - L + M - N),$$

or:

$$c_3 = \tfrac{17}{1008}(A - B + C - D) - \tfrac{1}{126}(E - F + G - H) - \tfrac{1}{504}(K - L + M - N),$$
$$c_5 = \tfrac{5}{504}(E - F + G - H) - \tfrac{1}{63}(A - B + C - D) + \tfrac{1}{126}(K - L + M - N).$$

But $$A - B + C - D = -\delta^4(z_{0,1} - z_{0,-1}),$$

while $$E - F + G - H = -\delta^2\delta'^2(z_{0,1} - z_{0,-1}),$$

and $K - L + M - N = -\delta^4\delta'^2(z_{0,1} - z_{0,-1}) - 4\delta^2\delta'^2(z_{0,1} - z_{0,-1}) - 2\delta^4(z_{0,1} - z_{0,-1})$.

Thus:

$$c_3 = -\tfrac{1}{48}\delta^4(z_{0,1} - z_{0,-1}) - \tfrac{1}{504}\delta^4\delta'^2(z_{0,1} - z_{0,-1}) \text{ ......(xxxvii)},$$
$$c_5 = -\tfrac{1}{24}\delta^2\delta'^2(z_{0,1} - z_{0,-1}) - \tfrac{1}{126}\delta^4\delta'^2(z_{0,1} - z_{0,-1}) \text{ ...(xxxviii)}.$$

Similarly:

$$c_4 = -\tfrac{1}{48}\delta'^4(z_{1,0} - z_{-1,0}) - \tfrac{1}{504}\delta^2\delta'^4(z_{1,0} - z_{-1,0}) \text{ ......(xxxix)},$$
$$c_6 = -\tfrac{1}{24}\delta^2\delta'^2(z_{1,0} - z_{-1,0}) - \tfrac{1}{126}\delta^2\delta'^4(z_{1,0} - z_{-1,0}) \text{ ....(xl)}.$$

Substituting in (xxxii) we find for our Mid-point Semi-Lagrangian Quintic:

$$
\begin{aligned}
z_{\theta,\chi} = z_{0,0} &+ \tfrac{1}{2}\chi\,(z_{0,1} - z_{0,-1}) + \tfrac{1}{2}\theta\,(z_{1,0} - z_{-1,0}) + \tfrac{1}{4}\theta\chi\,(z_{1,1} - z_{1,-1} - z_{-1,1} + z_{-1,-1}) \\
&\left.\begin{aligned}
&+ \tfrac{1}{2}\chi^2\,(1 - \tfrac{1}{2}\theta^2)\,\delta'^2 z_{0,0} + \tfrac{1}{2}\theta^2\,(1 - \tfrac{1}{2}\chi^2)\,\delta^2 z_{0,0} \\
&\qquad + \tfrac{1}{8}\theta^2\chi^2\,(\delta'^2 z_{1,0} + \delta'^2 z_{-1,0} + \delta^2 z_{0,1} + \delta^2 z_{0,-1})
\end{aligned}\right\}\text{2nd order} \\
&\left.\begin{aligned}
&+ \tfrac{1}{4}\theta^2\chi\,(\delta^2 z_{0,1} - \delta^2 z_{0,-1}) + \tfrac{1}{4}\theta\chi^2\,(\delta'^2 z_{1,0} - \delta'^2 z_{-1,0}) \\
&\qquad - \tfrac{1}{12}\chi\,(1 - \chi^2)\,(\delta'^2 z_{0,1} - \delta'^2 z_{0,-1}) - \tfrac{1}{12}\theta\,(1 - \theta^2)\,(\delta^2 z_{1,0} - \delta^2 z_{-1,0})
\end{aligned}\right\}\text{3rd order} \\
&\left.\begin{aligned}
&- \tfrac{1}{24}\theta\chi\,(1 - \chi^2)\,(\delta'^2 z_{1,1} - \delta'^2 z_{1,-1} - \delta'^2 z_{-1,1} + \delta'^2 z_{-1,-1}) \\
&\qquad - \tfrac{1}{24}\theta\chi\,(1 - \theta^2)\,(\delta^2 z_{1,1} - \delta^2 z_{1,-1} - \delta^2 z_{-1,1} + \delta^2 z_{-1,-1}) \\
&- \tfrac{1}{24}\theta^2\,(1 - \theta^2)\,\delta^4 z_{0,0} - \tfrac{1}{24}\chi^2\,(1 - \chi^2)\,\delta'^4 z_{0,0}
\end{aligned}\right\}\text{4th order} \\
&\left.\begin{aligned}
&- \tfrac{1}{48}\theta^2\chi\,(1 - \theta^2)\,(\delta^4 z_{0,1} - \delta^4 z_{0,-1}) - \tfrac{1}{48}\theta\chi^2\,(1 - \chi^2)\,(\delta'^4 z_{1,0} - \delta'^4 z_{-1,0}) \\
&+ \tfrac{1}{240}\theta\,(1 - \theta^2)\,(4 - \theta^2)\,(\delta^4 z_{1,0} - \delta^4 z_{-1,0}) \\
&\qquad + \tfrac{1}{240}\chi\,(1 - \chi^2)\,(4 - \chi^2)\,(\delta'^4 z_{0,1} - \delta'^4 z_{0,-1}) \\
&- \tfrac{1}{24}\theta\chi^2\,(1 - \theta^2)\,(\delta^2\delta'^2 z_{1,0} - \delta^2\delta'^2 z_{-1,0}) \\
&\qquad - \tfrac{1}{24}\theta^2\chi\,(1 - \chi^2)\,(\delta^2\delta'^2 z_{0,1} - \delta^2\delta'^2 z_{0,-1})
\end{aligned}\right\}\text{5th order} \\
&\left.\begin{aligned}
&- \tfrac{1}{108}\theta\chi\,\{(1 - \theta^2) + (1 - \chi^2)\}\,(\delta^2\delta'^2 z_{1,1} - \delta^2\delta'^2 z_{1,-1} \\
&\qquad - \delta^2\delta'^2 z_{-1,1} + \delta^2\delta'^2 z_{-1,-1})
\end{aligned}\right\}\text{6th order} \\
&\left.\begin{aligned}
&- \tfrac{1}{504}\theta^2\chi\,(1 - \theta^2)\,(\delta^4\delta'^2 z_{0,1} - \delta^4\delta'^2 z_{0,-1}) \\
&\qquad - \tfrac{1}{504}\theta\chi^2\,(1 - \chi^2)\,(\delta^2\delta'^4 z_{1,0} - \delta^2\delta'^4 z_{-1,0}) \\
&- \tfrac{1}{126}\theta\chi^2\,(1 - \theta^2)\,(\delta^2\delta'^4 z_{1,0} - \delta^2\delta'^4 z_{-1,0}) \\
&\qquad - \tfrac{1}{126}\theta^2\chi\,(1 - \chi^2)\,(\delta^4\delta'^2 z_{0,1} - \delta^4\delta'^2 z_{0,-1})
\end{aligned}\right\}\text{7th order} \\
&\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\text{(xli)}.
\end{aligned}
$$

Comparing with the result (xxvii) on p. 29 we see that the best fitting quintic has reproduced all the same fifth order terms as the fundamental mid-point formula. Or we conclude that up to fifth order terms the fundamental mid-point central difference formula not only passes through the 13 nearest points, but is itself the best fitting quintic to the next 16 nearest points. It is only in the sixth and seventh order terms that a difference in the coefficients appears. Thus unless we get beyond fifth order terms, it is not possible to say whether the Semi-Lagrangian Quintic is better or not than the fundamental mid-point central difference formula.

Comparison of Numerical Results.

Illustration (i). To apply the mid-point Semi-Lagrangian Quintic to the table on p. 30, we have simply to add the sixth and seventh order terms to the result ·885,1140 (24) of the terms of the mid-point central difference formula up to the fifth.

	Mid-point Semi-Lagrangian Quintic	*Fundamental Mid-point Formula*
Including 5th differences	·885,1140 (24)	·885,1140 (24)
,, 6th ,,	·885,1140 (26)	·885,1140 (24)
,, 7th ,,	·885,1140 (25)	·885,1139 (67)

The degree of accuracy of the table as calculated, i.e. to the seventh figure, does not enable us to determine which is the more accurate formula to the order of differences investigated. The mid-point Semi-Lagrangian Quintic appears somewhat steadier. I have worked out a second illustration where the original table is to nine figures to obtain further light on this point.

Illustration (ii). Find $F(p, u)$ for $p = \cdot 7$, $u = \cdot 9$, i.e. $\theta = -\frac{1}{2}$, $\chi = -\frac{1}{2}$, from the following data:

	$u = \cdot 6$	$u = \cdot 8$	$u = 1\cdot 0$
$p = \cdot 2$	·8570,20192	·8108,17523	·7662,34889
δ^2	− 1638395	− 1365891	− 1130504
δ'^2	1629207	1620035	1604047
δ^4	125494	109205	93896
δ'^4	− 7018	− 6816	− 6485
$\delta^2\delta'^2$	—	—	—
$p = \cdot 4$	$(z_{1,1})$ ·8361,95349	$(z_{1,0})$ ·7824,66908	$(z_{1,-1})$ ·7305,39359
δ^2	− 1723846	− 1454351	− 1217532
δ'^2	1797279	1800892	1796180
δ^4	62567	55244	47938
δ'^4	− 8266	− 8325	− 8207
$\delta^2\delta'^2$	− 33962	− 32676	− 31143
$p = \cdot 6$	$(z_{0,1})$ ·8136,46660	$(z_{0,0})$ ·7526,61942	$(z_{0,-1})$ ·6936,26297
δ^2	− 1746730	− 1487567	− 1256622
δ'^2	1931389	1949073	1957370
δ^4	33019	29898	26426
δ'^4	− 9002	− 9387	− 9546
$\delta^2\delta'^2$	− 28618	− 28218	− 27359
$p = \cdot 8$	$(z_{-1,1})$ ·7893,51241	$(z_{-1,0})$ ·7213,69409	$(z_{-1,-1})$ ·6554,56613
δ^2	− 1736595	− 1490885	− 1269286
δ'^2	2036881	2069036	2091201
δ^4	17735	16702	15211
δ'^4	− 9248	− 9990	− 10484
$\delta^2\delta'^2$	− 23891	− 24111	− 23889
$p = 1\cdot 0$	·7633,19227	·6885,85991	·6160,17643
δ^2	− 1708725	− 1477501	− 1266739
δ'^2	2118482	2164888	2201143
δ^4	9529	9559	9085
δ'^4	− 9087	− 10151	− 11006
$\delta^2\delta'^2$	—	—	—

The following are the results obtained:

Differences	Mid-point Interpolation General Formula	Semi-Lagrangian Quintic
1st order	·7056,17561 (6)	·7056,17561 (6)
2nd ,,	·7056,74808 (9)	·7056,74808 (9)
3rd ,,	·7056,98834 (1)	·7056,98834 (1)
4th ,,	·7056,98545 (7)	·7056,98545 (7)
5th ,,	·7056,98294 (4)	·7056,98294 (4)
[Here the two formulae start to differ and we find:]		
6th ,,	·7056,98304 (6)	·7056,98304 (9)
7th ,,	·7056,98278 (6)	·7056,98304 (2)

The computed value is ·7056,98337. Accordingly both formulae give results correct up to the seventh figure. But the Semi-Lagrangian Quintic is only three points wrong in the eighth figure, while the general mid-point central difference formula is six points wrong in the eighth figure*. Thus the Quintic gives a somewhat better result than the general formula and it is quite easy in this instance to trace the divergence to its source. The seventh order terms are in the case of the general formula negative and give $-26\,(0)$. The whole of this change (and more for the other terms are *positive*) arises from the term in the formula:

$$-\tfrac{1}{10080}\,\theta\,(1-\theta^2)\,(4-\theta^2)\,(9-\theta^2)\,(\delta^6 z_{1,0}-\delta^6 z_{-1,0}).$$

This term arises from our starting assumption on p. 26 that the univariate interpolation shall be a curve passing through nine points along the axis of x (i.e. of the argument p). To get the differences involved we need to use the table entry for $p=-\cdot 2$, and at such a point the influence of the border of the table at $p=-1\cdot 0$ is still appreciable. This is a ridge or boundary where certain of the differential coefficients of the function become infinite, and the rapidly changing δ^2, δ^4, δ^6 differences are indicative of this.

Now the Semi-Lagrangian Quintic uses only the nearest 29 points (see p. 43) or the points on the six nearest circles and does not involve the point at $p=-\cdot 2$, $u=\cdot 8$, i.e. δ^6 differences. This result therefore suggests that the principle we have already emphasised (see pp. 20, 24 and 40) may be of considerable importance. Namely, that it may be better to use formulae based

* The following are the additional differences required for the sixth and seventh order terms:

	δ^6	δ'^6	$\delta^4\delta'^2$	$\delta^2\delta'^4$
$z_{1,0}$	+28615	+177	+17	+447
$z_{-1,0}$	+6053	−248	−458	+442
$z_{0,1}$	+14264	+236	+617	+490
$z_{0,-1}$	+10297	+236	−114	+401

solely on next nearest points, rather than formulae which break down along our axes into uni-variate formulae with high degrees of axial approximation. If this conclusion be correct the Semi-Lagrangian Quintics investigated in this section of our tract may turn out to be the best available bi-variate central difference formulae.

Bi-variate Central Difference Boundary Formulae.

Just as in the case of uni-variate interpolation we may need bi-variate boundary formulae when the central differences only are provided in our table and where it is impossible to table δ'^2, δ'^4 in the panel next the boundary and δ'^4 in the panel once removed from the boundary. In the ordinary case δ^2 and δ^4 will, however, be given for both first and second panels. We thus need a 1st Panel-Ordinary and a 2nd Panel-Ordinary formula. We may need again to interpolate in the panels adjacent to a double boundary of the table. In such a case we require *three* formulae. Namely:

(i) 1st Panel—1st Panel Formula,

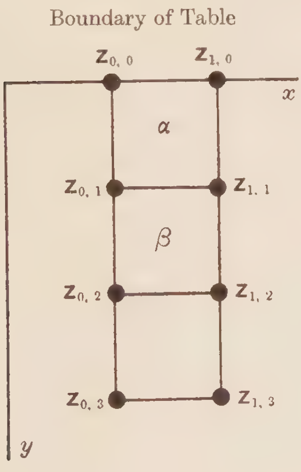
(ii) 1st Panel—2nd Panel Formula,

(iii) 2nd Panel—2nd Panel Formula.

Clearly a 2nd Panel—1st Panel Formula can be obtained from (ii) by interchange of variates and need not be specially provided.

The schemes are as follows:

1st Panel-Ordinary and 2nd Panel-Ordinary.

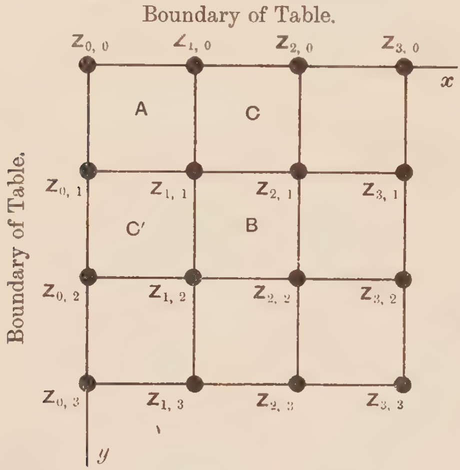


Known	*Known*
δ'^2,	δ^2, δ^4, δ^6
δ'^2, δ'^4	,, ,, ,,
δ'^2, δ'^4, δ'^6	,, ,, ,,

α is 1st Panel-Ordinary.
β is 2nd Panel-Ordinary.

4

While the formulae provided on the accompanying sheet may suffice in certain cases to interpolate at the borders of a table of double entry they do not meet the bulk of the cases wherein the function itself terminates at those boundaries. Where the boundaries are merely artificial, there it is often possible to calculate the central differences along the boundaries, and then these formulae are unnecessary. On the other hand where the boundaries form a real limit to



Known.

For $z_{1,1}$ δ^2 and δ'^2 are known
" $z_{2,2}$ δ^4 " δ'^4 " "
" $z_{3,3}$ δ^6 " δ'^6 " "

A is 1st Panel-1st Panel. *B* is 2nd Panel-2nd Panel. *C* and *C′* are 1st Panel-2nd Panel, and 2nd Panel-1st Panel respectively.

the function, the differential coefficients of the function may not be finite at the boundaries, the differences will then tend to diverge and any interpolation formulae based upon differences be unsatisfactory. Occasionally the difficulty is only met with in one set of differences, say x, and not in the other y. In this case our formulae may provide fairly satisfactory results in the immediate neighbourhood of a tabled column of y values, but will not be closely approximate elsewhere. Occasionally also an auxiliary function may be used, instead of the actual function in the neighbourhood of the boundaries, which does not suffer from diverging differences and to which the above formulae can be applied. But such auxiliary functions are not always easy of discovery. For example

$$F(x, y) = 1 + \frac{x(1+y)^{\frac{1}{2}}}{y+2} + \frac{x^2(1+y)}{(y+2)(y+3)} + \frac{x^3(1+y)^{\frac{3}{2}}}{(y+2)(y+3)(y+4)} + \dots$$

is a function having divergent differences in the region of y negative and less than unity, and for which it is not easy to find an apt auxiliary function for tabling by equal intervals of x and y, and admitting of ordinary interpolation.

The computer of tables is accordingly warned that in the construction of tables there may be grave difficulties in interpolation from a frame in what may be conveniently called "finial" regions.

On the Accuracy of Central Difference Interpolation.

Inaccuracy arises from three sources:

(i) From our differences ultimately diverging, as when the tabled function has one or more of its differential coefficients at certain points or along certain lines infinite. There is no remedy for this except the use of special tables involving auxiliary functions in the neighbourhood of such regions.

(ii) We have not taken an adequate number of differences. It is perfectly easy to test this for any formula or case. For example take (xi) the mid-panel central difference formula of p. 14 of Tract II, the maximum values of the last coefficients are about ·0025, actually ·002,452, if we go to sixth differences; and there will be two differences multiplied by coefficients of this order. Accordingly we need not include sixth differences, if they range under 100. Having found the order of our differences, and knowing the maximum values of our coefficients, it is always possible to retain such differences as will alone influence the last figure of our interpolate. There is no practical necessity to evaluate residuals with the rapid convergency of the central difference formulae coefficients.

(iii) The tabulated values may differ from the true values by something at a maximum slightly under half a unit in the last tabulated figure.

We can obtain a superior limit to this the most important source of error for the case of central differences by remembering that our mid-panel Central Difference formula is an even-point Lagrangian, and our mid-point Central Difference formula an odd-point Lagrangian. Now we have expressed these Lagrangians as linear functions of an even or an odd number of ordinates, and evaluated the numerical values of the coefficients of these ordinates for a large number of cases. All we have to do therefore is to treat the ordinates as equal to ·5, add the coefficients together as if they were all positive terms, and we shall reach the maximum error that can arise in the last figure of the interpolate from less than ·5 errors in the last figures of the interpolants. Before tabling the results reached in this manner we may remark on a point often overlooked by writers on interpolation, i.e. that a mid-panel central difference formula ought only to be used in the panel strip − ·25 to + ·25 from its centre, and mid-point central difference formula in the strip from − ·25 to + ·25 round its mid-ordinate; thus the two formulae are supplementary, like Newton's *Casus I* and *Casus II.*

Now we have not worked interpolation formulae at ·25 of the panel breadth, but only at the tenths. If, however, we find the errors for 0, ·1, ·2, ·3, we shall

have a due appreciation of what it is at ·25, where it is practically a maximum for the mid-point formula; it will be a maximum for 0 in the mid-panel formula.

Table of Errors of Central Difference Formulae.

Mid-Point Central Difference (I) or Odd-Point Lagrangian (II).

Distance from mid-point	(I) to 3rd order Diff. (II) through 5 points	(I) to 5th order Diff. (II) through 7 points	(I) to 7th order Diff. (II) through 9 points	(I) to 9th order Diff. (II) through 11 points
0	·500	·500	·500	·500
·1	·569	·584	·596	·606
·2	·623	·653	·675	·693
·3	·663	·703	·734	·758
·5	·695	·744	·782	·812
0	·625	·695	·744	·782
·1	·620	·687	·734	·769
·2	·605	·663	·703	·734
·3	·580	·623	·653	·675
·5	·500	·500	·500	·500
Distance from mid-panel	(III) to 3rd order Diff. (IV) through 4 points	(III) to 5th order Diff. (IV) through 6 points	(III) to 7th order Diff. (IV) through 8 points	(III) to 9th order Diff. (IV) through 10 points

Mid-Panel Central Difference (III) or Even-Point Lagrangian (IV).

It will be therefore clear that the maximum possible divergence of these formulae is about 0·6 to 0·7 in the last figure of the interpolate; the higher the difference used the higher the maximum. But this maximum is very far from the *probable* error likely to occur, especially in the high order difference formulae. In order that this maximum may occur all the tabulated entries for the n points would not only have to reach their maximum error, but all these errors would have to be of *definite* sign. For n large this is exceedingly improbable and the result shows, what experience confirms, that an error in the last figure of an interpolate is exceedingly unlikely to equal ·5, or with very few exceptions the interpolate is as accurate as the interpolants. In these exceptional cases its error will not exceed unity in the last figure of the value.

We can compare these errors with those of the forward difference formulae by the simple process of remembering that forward difference interpolation is merely central difference or Lagrangian formulae interpolation applied to an end-panel. Thus from Equations (α) (iii), (β) (iv), (γ) (ix), (δ) (x) of Tract II of this series we find at the mid-point of the panel when using forward differences for 3, 5, 7 and 9 differences maximum errors of 1·086, 2·132, 4·810

and 12·330*. These errors may not be often reached but they serve as comparative criteria, and show the great superiority of the central difference formulae.

It remains to consider how far errors of the uni-variate central difference formulae are increased when we pass to bi-variate central difference formulae. These formulae are far more troublesome to deal with as the general Lagrangians which have to be determined are very cumbersome. I have therefore confined myself to two special cases with the following results.

Mid-panel Bi-variate Difference Formula. Equation (iv), p. 8.

Proper area for application: $\theta = \frac{1}{2} \pm \frac{1}{4}$, $\chi = \frac{1}{2} \pm \frac{1}{4}$, up to and including fifth order differences.

$\theta = \phi = \chi = \psi = \frac{1}{2}$:	Maximum Error = ·922,
$\theta = \frac{1}{4}, \phi = \frac{3}{4}, \chi = \frac{1}{4}, \psi = \frac{3}{4}$:	„ „ = ·893,
$\theta = \frac{1}{4}, \phi = \frac{3}{4}, \chi = \psi = \frac{1}{2}$:	„ „ = ·865.

Mid-point Bi-variate Difference Formula. Equation (xxvii), p. 29.

Proper area for application: $\theta = 0 \pm \frac{1}{4}$, $\chi = 0 \pm \frac{1}{4}$, up to and including fifth order differences.

$\theta = 0, \chi = 0$:	Maximum Error = ·500,
$\theta = 0, \chi = \frac{1}{4}$:	„ „ = ·777,
$\theta = \frac{1}{4}, \chi = \frac{1}{4}$:	„ „ = ·903.

The reader who will compare these results with those for the uni-variate formulae on p. 52, comparing the latter for the fifth difference column, will see that at the limits of the proper field of application the maximum errors in the bi-variate formulae are about 1·3 to 1·4 times the maximum errors in the uni-variate formulae. The bi-variate formulae are thus unlikely to lead to an error greater than one in the last figure. Indeed the chance of an error of unity is, I think, very much smaller for the bi-variate than the uni-variate formulae. For to obtain an error of ·695 in a uni-variate result from a mid-panel formula, it would be needful for *seven* entries all simultaneously to have their maximum errors, and a definite, i.e. non-random system, of signs to exist for these errors; but to obtain an error of ·922 in the corresponding bi-variate results no less than *twenty-five* entries must take their maximum error values simultaneously, and these errors must in each case be of a definite sign†!

* Slightly larger maximum errors will be found at about ·4 instead of ·5 of the panel from the start of the series, but the above values are adequate to indicate the size of error.

† Thus the following must be of one sign: errors in $z_{0,3}$, $z_{-1,2}$, $z_{-2,1}$, $z_{0,1}$, $z_{1,1}$, $z_{-2,0}$, $z_{0,0}$, $z_{1,0}$, $z_{3,0}$, $z_{-1,-1}$, $z_{2,-1}$, $z_{0,-2}$, $z_{1,-2}$, but the errors in $z_{0,2}$, $z_{1,2}$, $z_{-1,1}$, $z_{2,1}$, $z_{-3,0}$, $z_{-1,0}$, $z_{2,0}$, $z_{-2,-1}$, $z_{0,-1}$, $z_{1,-1}$, $z_{-1,-2}$, $z_{0,3}$ must all be of the opposite sign.

It is, I think, safe to assert that if we interpolate into either a uni-variate or bi-variate table by central difference formulae with the adequate number of differences, there will hardly ever be an error introduced into the value of the interpolate to the same number of significant figures. In the construction of tables from frames, if the frame entries run to $(n + 1)$ figures and we proceed by Lagrangian or central difference interpolation to insert the values for a complete table we can cut down to n figures and be quite certain of the accuracy of our entries to the thus tabulated n figures.

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